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EXTENSIONS OF KRIPKE'S EMBEDDING THEOREM *

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Kripke's theorem on the embeddability of every Boolean algebra in a countably generated one is proved anew, supplemented by exact cardinality estimates and generalized to equational and other classes of partially ordered structures. Similar extensions of the Gaifman–Hales Theorem follow. The proofs use algebras of infinitary ($\mathcal{L}_{\infty\omega}$) formulas, and occasionally Lévy's theorem on Σ_1 formulas and the method of forcing.

Introduction

This work develops a method for embedding any given partially ordered structure (in particular, a Boolean algebra) in a structure of the same type which is generated (using possibly infinitary meets and joins) by a given number of generators and shares many properties in common with the given structure.

The basic construction can be described either in terms of infinitary ($\mathcal{L}_{\infty\omega}$) formulas (§ 1–4) or algebraically as the formation of a substructure of a direct power (§ 20–21). It is analyzed so as to give cardinality estimates and other data, so that many natural questions (e.g., determine the power of the free $<\kappa$-complete Boolean algebra or lattice on $\aleph_\alpha$ generators) are easily answered, and embedding theorems for lattices and Boolean algebras satisfying distributive or other laws are obtained.

Although a purely algebraic exposition of the results and proofs is possible, except for some results whose subject matter is partly non-algebraic (§ 12, § 27), such an exposition would be artificial at the res-

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ent stage of study. Our work evolved from a study of infinitary logic and this still seems the most natural way to present it, though the techniques used in the proofs underwent several changes.

The previously known results related to our subject (see [3, 7, 12, 22], [8, § 19] and also (15, 2.3]), all of which deal mainly with countably generated Boolean algebras, were for the most part discovered, or elegantly proved, by working with the Scott–Solovay version of Cohen's forcing method [20]. In § 25–26 we show the close relation between the Boolean algebras which our approach gives and those to which forcing considerations lead. It seems however, that at least our results concerning partially-ordered structures other than Boolean algebras are not directly accessible by current published methods.

The most important single idea which we owe to previous work, besides the basic results of Gaifman–Hales and Kripke, is the following observation, which goes back to Rasiowa and Sikorski at least in the case of finitary languages (cf. [18, Ch. VI, § 10–11]):

In the Lindenbaum algebra of a theory where each formula has only finitely many free variables, the equivalence class of $(\forall u)\ (\phi(u))$ is the meet of the equivalence classes of the substitution instances $\phi(\genfrac{}{}{0pt}{}{u}{v})$, $v \in V$, where V is the infinite set of all free variables. A dual statement holds for existential formulas $(\exists u)\ (\phi(u))$. This observation was used to prove the completeness of the predicate calculus.

So much for relations with previous work. We now sketch the contents of the present work in more detail: Chapter I (§ 1–18) concentrates on Boolean algebras (B.a.'s) (without distributive laws) and Chapter II treats the extension to partially ordered structures and some special topics. The starting point is a pair of well-known theorems on B.a.'s: The Gaifman–Hales theorem states that there are arbitrarily large countably generated B.a.'s, and Kripke's embedding theorem states that (moreover) every B.a. has a complete embedding in some countably generated one.

In § 1–4 we prove both these theorems. Short proofs have been given in the past, but the present new proofs are particularly suited for giving more detailed and more general information about embeddings. The main tools are Boolean-valued models and B.a.'s of formulas in the infinitary language $\mathcal{L}_{\infty\omega}$. (A reader who wants to know only the basic ideas of the work is advised to read § 1–4 and the first half of Ch. II.)

In §5 an analysis of the proofs is given, and the rest of Ch. I (except §11 which is partly expository and follows Gaifman's paper about infinite Boolean polynomials, and §18 to which we return below) is devoted to three main topics.

(a) §6–10 deal with problems of cardinality and complexity of generation in connection with countably generated B.a.'s and embeddings in them.

(b) §12–14 give theorems about derivability in infinitary languages, which are syntactical counterparts of (the Gaifman–Hales and) Kripke's theorem, and deduce new theorems about embeddings of free B.a.'s on many generators in free B.a.'s on $\aleph_0$ generators.

(c) §15–16 generalize most of the results of (a), (b) from $\aleph_0$ generators to $\aleph_\alpha$ generators for arbitrary α.

A subsidiary topic (§17 and elsewhere) is the determination of the maximal number of disjoint elements in free B.a.'s.

To give a better idea of the subject-matter we quote here some of the main results. Notations are mostly self-explanatory but are also defined in the Preliminaries below. For simplicity assume that κ is a regular cardinal. $\aleph_\alpha^{<\kappa} = \Sigma_{\nu<\kappa} \aleph^\nu$.

(1) The powers of infinite $(\leqslant \aleph_\alpha, <\kappa)$-generated B.a.'s are just all infinite cardinals $\lambda \leqslant \aleph_\alpha^{<\kappa}$. Every B.a. whose power is $\leqslant \aleph_\alpha^{<\kappa}$ (and no other B.a.) can be completely embedded in a $(\leqslant \aleph_\alpha, <\kappa)$-generated B.a. of the same power. [In §9–10 similar, somewhat less conclusive, results about *complete* countably generated B.a.'s are given. An exact characterization of the non-strongly-Mahlo powers of such B.a.'s is mentioned and partly proved in §9. The proof is completed in [25].]

(2) If $\lambda \leqslant \aleph_\alpha^{<\kappa}$ then $\mathcal{F}^{<\kappa}(\lambda)$ (the free $<\kappa$ complete B.a. on λ generators) has a $<\kappa$-complete embedding in $\mathcal{F}^{<\kappa}(\aleph_\alpha)$.

(3) $\mathcal{F}^{<\kappa}(\aleph_\alpha)$ has power $\aleph_\alpha^{<\kappa}$ and contains $2^{<\kappa}$, but not more, disjoint elements.

A somewhat surprising point about the development in Ch. I is the basic role (in §1–4) of the predicate language, since in a metamathematical treatment of B.a.'s one might expect the language of Boolean terms (i.e., the propositional language) to suffice. In §18 we show that one can indeed get the previous results without going beyond the propositional language, the key idea being the use of a theorem of A. Lévy about Σ_1 formulas of set theory. However, when one thinks of further developments, the two approaches (of § 1–4 and of § 18) are seen to be non-equivalent. The first leads to the results of Ch. II, which are only partly accessible by the second. The second is not yet thoroughly explored, but it has already led to a large body of refinements of the Gaifman–Hales theorem (see [26]).

About the contents of Ch. II we shall only say here that it shows how the main embedding results about B.a.'s carry over to other classes of partially-ordered structures, among them all equational classes. In some cases (pseudo-Boolean algebras for example) the results about free B.a.'s generalize too. The connection with forcing and the preservation of distributive laws in the embeddings are also treated. For more details the opening paragraphs of Ch. II should be consulted. The concluding section (§ 27) contains a brief list of some algebraic problems and areas of research suggested by the work, and also gives an example of an application of our results to logic and set theory.

An elementary knowledge of set theory, first-order logic, Boolean algebras and partially ordered sets suffices for following the arguments, except at a few points (§ 18, § 25–27) where we assume somewhat more than could conveniently be explained in the text, and give detailed references instead.

This paper is the first (main) part of the author's Ph.D. thesis, prepared at the Hebrew University of Jerusalem under the direction of Professor Haim Gaifman. I am greatly indebted to him for his interest in the work, encouragement, suggestions and criticisms and for the considerable amount of time he spent on my work. I am also indebted to Menachem Magidor and Saharon Shelah for their contributions to § 17 (which are described there) and for many interesting conversations.

Preliminaries

(A) *Set theory.* We assume an elementary acquaintance with set theory, including the axiom of foundation which allows proofs by induction and definitions by recursion on the $\in$-relation. $\Rightarrow$, $\Leftrightarrow$, $\forall$, $\exists$ are sometimes used as abbreviations for English expressions, but this use of $\forall$, $\exists$ is avoided when the predicate-language (with $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, $\exists$ as basic or defined logical constants) is under discussion. Our set-theoretic notations are for the most part standard. A set a is *transitive* when $(\forall x \in a)(x \subseteq a)$. $\alpha, \beta, \gamma, \delta$ (possibly with sub- or superscripts) always denote ordinals, and k, l, m, n natural numbers. $\alpha < \beta$ iff $\alpha \in \beta$. ω is the set of natural numbers. If a is a set of ordinals we let $\sup a = \bigcup a$ (the first ordinal $\geqslant \alpha$ for each $\alpha \in A$) and $\operatorname{Sup} a = \sup_{\alpha \in A}(\alpha + 1)$. For any sets a, b

$$a \sim b = \{x \in a \mid x \notin b\} ;$$

$\mathrm{TC}(a)$ = the smallest transitive set x such that $x \supseteq a$.

dom(f) is the *domain* of the function f, $f\,''\,a = \{f(x) \mid x \in a\}$ and $f \restriction a$ is the *restriction* of f to a.

$$\begin{pmatrix} a & b & \dots & z \\ a' & b' & \dots & z' \end{pmatrix}$$

is the function with domain $\{a, b, \dots, z\}$ such that $a \mapsto a'$, $b \mapsto b'$, etc. If τ is a set-theoretic term and ϕ a set-theoretic formula then $\langle \tau(x) \mid \phi(x) \rangle$ is the function with domain $\{x \mid \phi(x)\}$ that maps each x to $\tau(x)$. We let

$$(a, b) = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}, \qquad (a, b, c) = \begin{pmatrix} 0 & 1 & 2 \\ a & b & c \end{pmatrix} \quad \text{etc.}$$

Cardinals are initial ordinals. $\aleph_0 = \omega$. κ, λ, μ always denote infinite cardinals ($\aleph_\alpha$'s), while ν varies on all cardinals. The cardinality (power) of x is denoted by $|x|$. $|\mathrm{TC}(x)|$ is called the *hereditary cardinality of* x. $H(\kappa) = \{x \mid |\mathrm{TC}(x)| < \kappa\}$. ν^+ is the first cardinal $> \nu$. A *limit cardinal* is one of the form $\aleph_\alpha$, where α is a limit ordinal ($\alpha = \bigcup \alpha > 0$). The *cofinality* $\mathrm{cf}(\alpha)$ is defined as the smallest β such that for some $f: \beta \to \alpha$, $\alpha = \sup(\mathrm{range}(f))$. If $|I| < \mathrm{cf}(\kappa)$ and $|a_i| < \kappa$ for each $i \in I$, then $|\bigcup_{i \in I} a_i| < \kappa$. κ is called *regular* when $\mathrm{cf}(\kappa) = \kappa$, otherwise singular.

(B) *Boolean algebras*. The operations of a Boolean algebra (B.a.) $\mathcal{B}$ are denoted by $\neg, \wedge, \vee$ (usually a superscript $\mathcal{B}$ is added). By definition

$$a \to^{\mathcal{B}} b = \neg^{\mathcal{B}} a \vee^{\mathcal{B}} b, \qquad a \leqslant^{\mathcal{B}} b \text{ iff } a \vee^{\mathcal{B}} b = b .$$

If $A \subseteq \mathcal{B}$ then $\bigwedge^{\mathcal{B}} A$ is the *meet* (g.l.b.) of A in $\mathcal{B}$, if such exists. Similarly $\bigvee^{\mathcal{B}} A$ is the *join* of A. Subalgebras are often identified with their underlying sets. A *$<\kappa$-subalgebra of* $\mathcal{B}$ is a subalgebra $\mathcal{C}$ such that for all $X \subseteq \mathcal{C}$, if $|X| < \kappa$ and x is the meet or join of X in $\mathcal{B}$, then $x \in \mathcal{C}$. We allow also $\kappa = \infty$ (a $<\infty$-subalgebra is usually called a *complete subalgebra*).

A *$<\kappa$-homomorphism* $h : \mathcal{B}_1 \to \mathcal{B}_2$ is a homomorphism that preserves meets and joins (existing in $\mathcal{B}_1$) of less than κ elements. A $<\infty$-homomorphism is called a *complete homomorphism*. A 1–1 homomorphism is called an *embedding*. $\mathcal{B}$ is said to be *$<\kappa$-complete* if every $X \subseteq \mathcal{B}$ of power $<\kappa$ has a meet and a join in $\mathcal{B}$. $\mathcal{B}$ is *complete* when it is $<\infty$-complete. The three "$<\kappa$" concepts are uninteresting for singular κ, because then "$<\kappa$" implies "$<\kappa^+$" assuming $\mathcal{B}$ ($\mathcal{B}_1$) is $<\kappa$-complete.

If $A \subseteq \mathcal{B}$ and κ is regular or $\kappa = \infty$, we let $[A]^{<\kappa}$ be the smallest $<\kappa$-subalgebra of $\mathcal{B}$ that contains A. For singular κ we put $[A]^{<\kappa}_{\mathcal{B}} = \bigcup_{\lambda<\kappa} [A]^{<\lambda^+}_{\mathcal{B}}$ (this equation is also true for regular $\kappa > \aleph_0$), and thus avoid the trivilization of our results for singular cardinals. There are reasons to regard the above definition as natural (even though $[A]^{<\kappa}_{\mathcal{B}}$ is not always a $<\kappa$-subalgebra of $\mathcal{B}$ when κ is singular) and it fits well with our theorems.

If $\mathcal{B} = [A]^{<\kappa}_{\mathcal{B}}$ we say that *A generates $\mathcal{B}$ in the $<\kappa$-sense* (including $\kappa = \infty$ in which case we simply say: *A generates $\mathcal{B}$*). $\mathcal{B}$ is *$(\nu, <\kappa)$-generated* when some $A \subseteq \mathcal{B}$ of power ν generates $\mathcal{B}$ in the $<\kappa$-sense. The definitions of *$(\leqslant\nu\ <\kappa)$-generated* and *$(<\nu, <\kappa)$-generated* are obtained by replacing "power ν" by "power $\leqslant\nu$ ($<\nu$)". When $\kappa = \infty$ we say simply that $\mathcal{B}$ is *$\nu(\leqslant\nu, <\nu)$-generated*. $\mathcal{B}$ is *countably-generated* when it is $\leqslant\aleph_0$-generated. It is easy to see that a $<\aleph_0$-generated B.a. is finite. On the other hand, $\aleph_0$-generated B.a.'s may be surprisingly large and rich in structure, as the following two basic theorems (which will be proved below) show.

Gaifman–Hales Theorem [3, 7, 22]. *There are countably generated B.a.'s of arbitrarily large powers.*

Kripke's embedding theorem [12]. *Every B.a. $\mathcal{B}$ has a complete embedding in a countably-generated B.a. $\mathcal{C}$.*

We assume some standard construction of the normal completion of a B.a. (e.g., completion by cuts), such that if $\mathcal{B}'$ is the normal completion of $\mathcal{B}$ then $\mathcal{B}'$ is a complete B.a. and $\mathcal{B}$ is a dense subalgebra of $\mathcal{B}'$ (that is, $(\forall x \in \mathcal{B}')\,[x \neq 0 \Rightarrow (\exists y \in \mathcal{B})\,(0 < y \leqslant x)]$). It follows that $\mathcal{B}$ is a regular subalgebra of $\mathcal{B}'$, i.e., the inclusion embedding $\mathcal{B} \subseteq \mathcal{B}'$ is complete. Also, $\mathcal{B}$ generates $\mathcal{B}'$ in the $<\infty$-sense, because if $x \in \mathcal{B}'$ then $x = \bigvee\{y \in \mathcal{B} \mid y \leqslant x\}$. Therefore, if $\mathcal{B}$ is countably generated, so is $\mathcal{B}'$, hence the countably generated B.a.'s mentioned in the above theorems may be taken as complete B.a.'s.

CHAPTER I

ON COUNTABLY GENERATED BOOLEAN ALGEBRAS

§1. A new proof of the Gaifman–Hales theorem

This section will show that the Gaifman–Hales theorem is an almost immediate consequence of some very basic facts concerning the infinitary language $\mathcal{L}_{\infty\omega}$. The idea of the proof, in more developed forms, underlies much of the present work.

Consider the language $\mathcal{L}$ with one binary predicate $\prec$. For any set A let $\mathcal{L}(A)$ be obtained from $\mathcal{L}$ by adding a name $\hat{a}$ for each $a \in A$. A model M for $\mathcal{L}$ consists of a domain of individuals $A \neq \emptyset$ (usually denoted by M) and a binary relation $\prec^M$ on it. We agree that for $a \in M$, $\hat{a}$ is automatically interpreted as a.

Atomic formulas of $\mathcal{L}(A)$ are of the form $t \prec t'$, where each of t and t' is a term of $\mathcal{L}(A)$, i.e., a variable or a name of an element of A. We take the variables to be u_α, v_α for all ordinals α, but only the u_α's are bindable by quantifiers. Formulas of $\mathcal{L}_{\infty\omega}(A)$ (or $\mathcal{L}_{\infty\omega}$ if $A = \emptyset$) are constructed by the following rules.

1.1. (1) Every atomic formula is a formula;

(2) if ϕ is a formula then $\neg\phi$ is a formula;

(3) if X is a set of formulas, then $\bigwedge X$ and $\bigvee X$ are formulas;

(4) if ϕ is a formula and u a bindable variable, then $(\forall u)\phi$ and $(\exists u)\phi$ are formulas.

$\wedge, \vee, \rightarrow, \leftrightarrow$ are defined in terms of $\neg$, $\bigwedge$, $\bigvee$ as usual. $\phi(^w_t)$ denotes the result of substituting the term t for the free occurrences of the variable w in the formula ϕ. When w is v_0, we shall write simply $\phi(t)$. If f is a function from a set of variables into A, we let $\phi[f]$ be the formula resulting from ϕ by replacing all free occurrences of each $w \in \mathrm{dom}(f)$ by the name of $f(w)$.

In a model M a truth-value is assigned to each $\mathcal{L}_{\infty\omega}(M)$ sentence in the natural way (taking $\hat{a}_1 \prec \hat{a}_2$ as true or false according as $a_1 \prec^M a_2$ or not). $M \models \phi$ means that ϕ is (an $\mathcal{L}_{\infty\omega}(M)$-sentence) true in M. $M \models \phi[f]$ means accordingly that $\phi[f]$ is true in M and is read "ϕ is satisfied in M by the assignment f".

Define by recursion on α the formula $\pi_\alpha = \pi_\alpha(v_0)$,

$$\pi_\alpha(v_0) = (\forall u)\,(u \prec v_0 \rightarrow \bigvee_{\beta<\alpha} \pi_\beta(u)) \wedge \bigwedge_{\beta<\alpha} (\exists u)\,(u \prec v_0 \wedge \pi_\beta(u)),$$

where $u = u_\alpha$ (so that u is not bound in π_β for any $\beta < \alpha$). It is obvious by induction on α that if M is a linearly ordered model, $a \in M$, then $M \models \pi_\alpha(\hat{a})$ iff a occupies the α^{th} place in $\prec^M$. In particular, let $<_\delta$ be the natural ordering of the ordinal δ. Then $\alpha < \delta$, $\alpha \neq \beta$, implies

$$\langle \delta, <_\delta \rangle \models (\pi_\alpha \wedge \neg \pi_\beta) \begin{bmatrix} v_0 \\ \alpha \end{bmatrix}.$$

Given two $\mathcal{L}_{\infty\omega}$-formulas ϕ, ψ, we say that ϕ *logically implies* ψ, and write (temporarily) $\phi \overset{*}{\rightarrow} \psi$, when for every model M and assignment f with values in M (defined at least for the free variables of ϕ and of ψ), $M \models \phi[f]$ implies $M \models \psi[f]$. Thus, the above shows that for $\alpha \neq \beta$, $\pi_\alpha \overset{*}{\nrightarrow} \pi_\beta$.

In preparation for the construction of a Boolean algebra, we list the following properties of $\overset{*}{\rightarrow}$, using ϕ, χ, ψ, u, v as syntactical variables with the obvious ranges, and letting X vary on arbitrary sets of $\mathcal{L}_{\infty\omega}$-formulas (in general, we have defined $\overset{*}{\rightarrow}$ only as a relation between $\mathcal{L}_{\infty\omega}$-formulas, i.e., formulas that do not contain names $\hat{a}$). In reading 1.2 keep in mind the definition of a B.a. as a complemented distributive lattice.

1.2. (1) $\phi \overset{*}{\rightarrow} \phi$ (reflexivity);

(2) if $\phi \overset{*}{\rightarrow} \chi$ and $\chi \overset{*}{\rightarrow} \psi$, then $\phi \overset{*}{\rightarrow} \psi$ (transitivity);

(3) if $\phi \in X$, then $\bigwedge X \overset{*}{\rightarrow} \phi$ (lower bound);

(4) if $\phi \overset{*}{\rightarrow} \psi$ for all $\psi \in X$, then $\phi \overset{*}{\rightarrow} \bigwedge X$ (greatest lower bound);

(5) if $\phi \in X$, then $\phi \overset{*}{\rightarrow} \bigvee X$ (upper bound);

(6) if $\psi \overset{*}{\rightarrow} \phi$ for all $\psi \in X$, then $\bigvee X \overset{*}{\rightarrow} \phi$ (least upper bound)

(7) $\phi \wedge (\chi \vee \psi) \overset{*}{\rightarrow} (\phi \wedge \chi) \vee (\phi \wedge \psi)$ (distributivity);

(8) $\phi \wedge \neg\phi \overset{*}{\rightarrow} \psi$ and $\psi \overset{*}{\rightarrow} \phi \vee \neg\phi$ (complementation).

1.3. (1) $(\forall u)(\phi) \overset{*}{\to} \phi(^u_v)$;
(2) if $\phi \overset{*}{\to} \psi(^u_v)$ and v is neither free in ϕ nor in ψ, then $\phi \overset{*}{\to} (\forall u)(\psi)$;
(3) $\phi(^u_v) \overset{*}{\to} (\exists u)(\phi)$;
(4) if $\psi(^u_v) \overset{*}{\to} \phi$ with v as in (2), then $(\exists u)(\psi) \overset{*}{\to} \phi$.

Now let κ be an infinite cardinal, and let T be the set of all $\mathcal{L}_{\infty\omega}$-formulas ϕ such that:

(a) ϕ has all its bound variables among $\{u_\alpha \mid \alpha < \kappa\}$, and $\bigwedge$, $\bigvee$ are applied only to sets of less than κ formulas in the formation of ϕ;

(b) the set of free variables of ϕ is a finite subset of $\{v_n \mid n < \omega\}$.

Clearly T is a set of formulas, closed under $\neg, \wedge, \vee$ and containing π_α for $\alpha < \kappa$. Now $\{(\phi, \psi) \mid \phi, \psi \in T, \phi \overset{*}{\leftrightarrow} \psi\}$ (where $\phi \overset{*}{\leftrightarrow} \psi$ is short for $\phi \overset{*}{\to} \psi$ and $\psi \overset{*}{\to} \phi$) is an equivalence relation, $\equiv$, on T. Denote by $[\phi]$ the $\equiv$-equivalence class of ϕ, for $\phi \in T$, and let $(T/{\equiv}) = \{[\phi] \mid \phi \in T\}$.

By 1.2, $T/{\equiv}$ can clearly be considered as a B.a. $\mathfrak{B}$, such that for all $\phi, \psi \in T$ the following hold (note that if $\bigwedge X$ or $\bigvee X \in T$ then $X \subseteq T$):

(I) $[\phi] \leqslant^{\mathfrak{B}} [\psi]$ iff $\phi \overset{*}{\to} \psi$;

(II) $[\neg\psi] = \neg^{\mathfrak{B}} [\psi]$;

(III) if $\phi = \bigwedge X$ then $[\phi] = \bigwedge^{\mathfrak{B}}_{\chi \in X} [\chi]$, and dually for $\bigvee X$.

Now let $\phi = (\forall u)(\phi' \in T)$. Then $\phi'(^{u}_{v_n}) \in T$ for each $n < \omega$. By 1.3, $\phi \overset{*}{\to} \phi'(^{u}_{v_n})$. If $\chi \in T$ and $\chi \overset{*}{\to} \phi'(^{u}_{v_n})$ for each n, then choosing n so that v_n is free neither in χ nor ϕ (which is possible because χ and ϕ have only finitely many free variables), we get $\chi \overset{*}{\to} \phi$ by 1.3. In terms of the B.a. $\mathfrak{B}$ this means:

(IV) if $\phi = (\forall u)(\phi')$ then $[\phi] = \bigwedge^{\mathfrak{B}}_n [\phi'(^{u}_{v_n})]$, and dually for $(\exists u)(\phi')$.

1.4. **Definition.** The *depth* $d(\phi)$ of a formula ϕ is defined recursively:

(1) $d(\phi) = 0$ if ϕ is atomic;

(2) $d(\phi) = d(\psi) + 1$ if $\phi = \neg\psi$ or $\phi = (\forall u)(\psi)$ or $\phi = (\exists u)(\psi)$;

(3) $d(\phi) = \mathrm{Sup}_{\psi \in X}\, d(\psi)$ $(= \sup_{\psi \in X}(d(\psi) + 1))$ if $\phi = \bigwedge X$ or $\phi = \bigvee X$.

Note that if $\phi = (\forall u)(\phi')$ or $\phi = (\exists u)(\phi')$ then $d(\phi) = d(\phi'(^u_t)) + 1$, because substitution preserves depth (t may be any term).

Let $B_\alpha = \{[\phi] \mid \phi \in T, d(\phi) = \alpha\}$. Then $\mathfrak{B} = \bigcup_\alpha B_\alpha$. By (I)–(IV) every element of B_α is obtained, if $\alpha > 0$, from elements of $\bigcup_{\beta<\alpha} B_\beta$ by $\neg^{\mathfrak{B}}$, $\bigwedge^{\mathfrak{B}}$ or $\bigvee^{\mathfrak{B}}$. Hence $\mathfrak{B}$ is generated by $B_0 = \{[\phi] \mid \phi \in T, \phi \text{ atomic}\} = \{[v_m < v_n] \mid m, n < \omega\}$. Thus $\mathfrak{B}$ is countably generated, and contains the κ different elements $[\pi_\alpha]$, $\alpha < \kappa$ (for $\alpha \neq \beta \Rightarrow \pi_\alpha \not\overset{*}{\to} \pi_\beta$). Since κ is arbitrary, this proves the Gaifman–Hales theorem.

It is useful to note that the proof could be carried out under other choices of the relation $\overset{*}{\to}$. One might take $\phi \overset{*}{\to} \psi$ to mean that $\phi \to \psi$ is Boolean valid, i.e., is provable in one of the standard proof systems for $\mathcal{L}_{\infty\omega}$. This relation will occupy a central place in later sections. Another possibility is to choose κ first, and then to let $\phi \overset{*}{\to} \psi$ mean that $\phi \to \psi$ is valid (holds under all assignments) in the model $\langle \kappa, <_\kappa \rangle$. Again 1.2 and 1.3 hold and the B.a. $\mathfrak{B}$ can be defined just as before. Since $\pi_\alpha(\hat{\gamma}) \wedge \pi_\beta(\hat{\gamma})$ $(\gamma < \kappa)$ is true in $\langle \kappa, <_\kappa \rangle$ iff $\alpha = \beta = \gamma$, it is clear that if $\alpha, \beta < \kappa$, $\alpha \neq \beta$, then $[\pi_\alpha]$ and $[\pi_\beta]$ are disjoint non-zero elements of $\mathfrak{B}$ (for the last choice of $\overset{*}{\to}$), so that $\mathfrak{B}$ contains κ disjoint elements. Actually $[\pi_\alpha]$, $\alpha < \kappa$ are pairwise disjoint for the previous choices of $\overset{*}{\to}$ as well, but we shall not dwell on this here.

§2. From ordinals to arbitrary sets

In §1 we have seen how in a model $\langle \delta, <_\delta \rangle$ each element $\alpha < \delta$ is definable by a certain (infinitary) formula $\pi_\alpha(v_0)$. The same situation appears in models of the form $\langle A, \in_A \rangle$ where A is a transitive set ($\neq \emptyset$). Each $x \in A$ can be "pinpointed" in such a model by a formula $\pi_x(v_0)$. Of course the same property of "$\mathcal{L}_{\infty\omega}$-pointwise definability" is enjoyed by any extensional well-founded model (being isomorphic to a transitive $\in$-model). In defining the formulas π_x, which we call locating (or pinpointing) formulas, we shall denote the binary predicate of the language by 'ϵ' instead of '$\prec$'.

2.1. Definition. Define the formula π_x by $\in$-recursion on x:

$$\pi_x(v_0) = (\forall u)\,(u \epsilon v_0 \to \bigvee_{y \in x} \pi_y(u)) \wedge$$

$$\wedge \bigwedge_{y \in x} (\exists u)\,(u \epsilon v_0 \wedge \pi_y(u)),$$

where u is the first bindable variable not bound in π_y for any $y \in x$. (A simple induction shows that this variable is $u = u_{\mathrm{rank}(x)}$ where rank(x) is the $\in$-rank of x, defined by rank$(x) = \mathrm{Sup}_{y \in x}$ rank(y).)

Putting $x = \alpha$, and writing $\prec$ for ϵ, we come back to the π_α of §1.

2.2. Lemma. *If A is transitive, then for all x and all $a \in A$,*

$$\langle A, \in_A \rangle \models \pi_x(\hat{a}) \quad \textit{iff} \quad x = a.$$

The proof, by $\in$-induction on y, is left to the reader.

It follows that the formulas π_x are pairwise incomparable in the sense that $x \neq y$ implies $\pi_x \not\overset{*}{\to} \pi_y$, where $\overset{*}{\to}$ is either logical implication or implication in $\langle A, \in_A \rangle$ for any transitive set A containing x, because then $\langle A, \in_A \rangle = (\pi_x \wedge \neg\pi_y) [\,{}^{v_0}_{x}\,]$ if $y \neq x$. (π_x and π_y are even mutually contradictory if $x \neq y$.)

§3. Boolean models and independent sentences

We have seen how to find, in the language based on one binary predicate ϵ, a proper class of pairwise non-logically equivalent formulas ($\pi_\alpha(v_0)$ for all α). It is just as easy to find non-equivalent sentences, for instance $(\exists u)\,(\pi_\alpha(u))\;(u = u_{\alpha+1})$ for all α. Indeed if $\alpha < \beta$, $\delta = \alpha + 1$, then $\langle \delta, <_\delta \rangle = (\exists u_{\alpha+1})\,(\pi_\alpha(u_{\alpha+1})) \wedge \neg(\exists u_{\beta+1})\,(\pi_\beta(u_{\beta+1}))$. But if we add a unary predicate P to the language $\mathcal{L}$, we can go one step forward, and find a proper class of $\mathcal{L}_{\infty\omega}$-sentences that are totally independent in the sense that in suitable models each subset of the class can realize any given distribution of truth values.

So let L be the language with the predicates ϵ and P. (The adaption of the syntactic and semantic notions of §1 to this new language is obvious.) If we were allowed to use names, we could offer the sentences $P(\hat{x})$ (all x) as an example of totally independent sentences. But since, in transitive $\in$-models, each x can be pinpointed by π_x, we can do without names, replacing $P(\hat{x})$ by $\rho_x = (\exists u)\,(\pi_x(u) \wedge P(u))$ (or by $(\forall u)\,(\pi_x(u) \to P(u))$; here $u = u_{\mathrm{rank}(x)+1}$, so that u is not bound in π_x).

3.1. Lemma. *Let U be a set. Then there is a model, in fact a transitive $\in$-model, for $\mathcal{L}$, in which ρ_x is true for $x \in U$ and false for $x \notin U$.*

Proof. Let A be a transitive non-empty set, $A \supseteq U$. Consider the model $M = \langle A, \in_A, U \rangle$ (i.e., P is interpreted by U). Then, by the property of

π_x as a locating formula for x in transitive $\in$-models, we get: $M \models \rho_x$ iff for some $a \in A$, $M \models \pi_x(\hat{a}) \wedge P(\hat{a})$ iff for some $a \in A$, $a = x$ and $a \in U$ iff $x \in U$. □

For the proof of Kripke's embedding theorem we shall want to know the independence of the ρ_x's in a stronger, Boolean, sense.

Let $\mathcal{B}$ be a complete B.a. The idea of $\mathcal{B}$-valued models is to regard $\mathcal{B}$ as the domain of truth values, with $1^{\mathcal{B}}$ corresponding to complete truth and other members of $\mathcal{B}$ to various extents of truth (and falsehood). Given a set A, a $\mathcal{B}$-valued n-place relation on A is just a function $R : A^n \to \mathcal{B}$. $R(\langle a_i \mid i < n \rangle)$ is the extent to which $\langle a_i \mid i < n \rangle$ falls under R. When $\mathcal{B}$ is taken as the two-element B.a. $\{0 \leqslant 1\}$, which we denote by $\bar{2}$, $\mathcal{B}$-valued relations are just characteristic functions of ordinary relations.

A $\mathcal{B}$-valued model M for $\mathcal{L}$ is a non-empty set (domain of individuals) A together with two $\mathcal{B}$-valued relations ϵ^M and P^M on A, i.e., $\epsilon^M : A^2 \to \mathcal{B}$, $P^M : A \to \mathcal{B}$. Then each $\mathcal{L}_{\infty\omega}(A)$-sentence is assigned a value in $\mathcal{B}$ in the following way:

3.2. (1) $\|\hat{a} \,\epsilon\, \hat{b}\|_M = \epsilon^M(a, b)$, $\|P(\hat{a})\|_M = P^M(a)$, $(a, b \in A)$;

(2) $\|\phi\|_M = \neg^{\mathcal{B}} \|\psi\|_m$ if $\phi = \neg\psi$;

(3) $\|\phi\|_M = \bigwedge_{\psi \in X} \|\psi\|_M$ if $\phi = \bigwedge X$, dually for $\bigvee X$;

(4) $\|\phi\|_M = \bigwedge_{a \in A} \|\psi(\tfrac{u}{\hat{a}})\|_M$ if $\phi = (\forall u)(\psi)$, dually for $(\exists u)(\psi)$.

The domain of individuals is usually denoted by M too. If it is a transitive set, and $\epsilon^M(a, b) = 1^{\mathcal{B}}$ if $a \in b$, $0^{\mathcal{B}}$ if $a \notin b$ (for $a, b \in M$), we call M a $\mathcal{B}$-valued transitive $\in$-model. Ordinary models are also called two-valued and stand in a 1–1 correspondence with $\bar{2}$-valued models.

3.3. Lemma. *Let M be a $\mathcal{B}$-valued transitive $\in$-model, x a set and $a \in M$. Then*

$$\|\pi_x(\hat{a})\|_M = \begin{cases} 1^{\mathcal{B}} & \text{if } a = x, \\ 0^{\mathcal{B}} & \text{if } a \neq x. \end{cases}$$

This is proved by $\in$-induction on x, just as in the corresponding proof for two-valued models. Indeed only $1^{\mathcal{B}}$ and $0^{\mathcal{B}}$ figure as truth values in the proof.

3.4. Theorem. *Let $\mathcal{B}$ be a complete Boolean algebra, and I a function into $\mathcal{B}$. Then there is a $\mathcal{B}$-valued model M such that $\|\rho_x\|_M = I(x)$ for all $x \in \mathrm{dom}(I)$ (and $\|\rho_x\|_M = 0$ for $x \notin \mathrm{dom}(I)$). Here we have $\rho_x = (\exists u)\,(\pi_x(u) \wedge P(u))\,(u = u_{\mathrm{rank}(x)+1})$, and π_x as defined in 2.1.*

Proof. Let A be a transitive non-empty set, $A \supseteq \mathrm{dom}(I)$, and consider the $\mathcal{B}$-valued model $M = \langle A, \in^M, P^M \rangle$ where, for $a, b \in A$,

$$\in^M(a,b) = \begin{cases} 1^{\mathcal{B}} & \text{if } a \in b, \\ 0^{\mathcal{B}} & \text{otherwise}, \end{cases} \qquad P^M(a) = \begin{cases} I(a) & \text{if } a \in \mathrm{dom}(I), \\ 0^{\mathcal{B}} & \text{otherwise}. \end{cases}$$

Then, for any x,

$$\|\rho_x\|_M = \bigvee_{a \in A}^{\mathcal{B}} (\|\pi_x(\hat{a})\|_M \wedge^{\mathcal{B}} \|P(\hat{a})\|_M)\,.$$

By Lemma 3.3,

$$\|\rho_x\|_M = \bigvee{}^{\mathcal{B}} \{\|P(\hat{a})\|_M \mid a \in A,\ a = x\}\ ,$$

hence

$$\|\rho_x\|_M = \begin{cases} 0^{\mathcal{B}} & \text{if } x \notin A \text{ or } x \notin A \sim \mathrm{dom}(I), \\ I(x) & \text{if } x \in \mathrm{dom}(I). \ \square \end{cases}$$

3.5. Remark. Theorem 3.4 is the Boolean generalization of Lemma 3.1, and gives the independence of the ρ_x's in the Boolean sense.

§4. A proof of Kripke's embedding theorem

Let $\mathcal{B}_0$ be a B.a. (which we shall embed in a countably generated one), and let $\mathcal{B}_1$ be its normal completion. Using Theorem 3.4 with $\mathcal{B}$ as $\mathcal{B}_1$ and I any function such that $\mathcal{B}_0 \subseteq \mathrm{range}(I) \subseteq \mathcal{B}_1$, we obtain a model M for $\mathcal{L}$ (the language with $\in$ and P) in which every $b \in \mathcal{B}_0$ is the value of some $\mathcal{L}_{\infty\omega}$-sentence. Define the set T of formulas as in the paragraph following 1.3, choosing κ large enough so that:

(*) Each $b \in \mathcal{B}_0$ has the form $\|\phi\|_M$ for some sentence $\phi \in T$.

Define the relation $\xrightarrow{*}$ between $\mathcal{L}_{\infty\omega}$-formulas by: $\phi \xrightarrow{*} \psi$ iff ϕ implies ψ in M, i.e., $\|\phi[f]\|_M \leqslant^{\mathcal{B}_1} \|\psi[f]\|_M$ for all assignments f into M It is easy to check that 1.2 and 1.3 hold for this $\xrightarrow{*}$ (for 1.3, a lemma about the effect of substitution, which has already been used implicitly several times, is needed). The main reason for choosing this relation $\overset{*}{\to}$ is that for $\mathcal{L}_{\infty\omega}$-*sentences* ϕ, ψ we get

$$\phi \overset{*}{\to} \psi \quad \text{iff} \quad \|\phi\|_M \leqslant \|\psi\|_M ,$$

hence

$$\phi \overset{*}{\leftrightarrow} \psi \quad \text{iff} \quad \|\phi\|_M = \|\psi\|_M .$$

Define now the relation $\equiv$ on T, the equivalence classes $[\phi]$ ($\phi \in T$) and the "algebra of formulas" $\mathcal{B}$ just as in § 1, and prove (I)–(IV) stated there again. Deduce that $\mathcal{B}$ is generated by

$$\{[\phi] \mid \phi \in T, d(\phi) = 0\} = \{[v_m \in v_n] \mid m,n < \omega\} \cup \{P(v_n) \mid n < \omega\},$$

hence is countably generated.

Let $S = \{\phi \in T \mid \phi$ is a sentence, $\|\phi\|_M \in \mathcal{B}_0\}$. By (*), $\mathcal{B}_0 = \{\|\phi\|_M \mid \phi \in S\}$ and we know that for $\phi, \psi \in S$ $[\phi] = [\psi]$ iff $\phi \overset{*}{\leftrightarrow} \psi$ iff $\|\phi\|_M = \|\psi\|_M$. Hence the equation $F(\|\phi\|_M) = [\phi]$ ($\phi \in S$) defines a (single-valued) 1 – 1 function $F\colon \mathcal{B}_0 \to \mathcal{B}$. S is closed under $\neg, \wedge, \vee$ and, for $\phi \in S$,

$$F(\neg \|\phi\|_M) = F(\|\neg\phi\|_M) = [\neg\phi] = \neg[\phi],$$

so that F preserves $\neg$ and similarly $\wedge, \vee$. We shall show that F is a complete embedding of $\mathcal{B}_0$ in $\mathcal{B}$ by proving that it preserves meets (and, dually, joins). Let $A \subseteq \mathcal{B}_0$, $a = \bigwedge^{\mathcal{B}_0} A$. Since $\mathcal{B}_0 = \{\|\phi\|_M \mid \phi \in S\}$, there exist $\phi \in S$, $X \subseteq S$ such that $a = \|\phi\|_M$ and $A = \{\|\psi\|_M \mid \psi \in X\}$. $\mathcal{B}_0$ is a regular subalgebra of $\mathcal{B}_1$, so $a = \bigwedge^{\mathcal{B}_1} A$, hence $\|\phi\|_M = a = \|\bigwedge X\|_M$ (though maybe $\bigwedge X \notin T$), and thus $\phi \overset{*}{\to} \bigwedge X \overset{*}{\to} \phi$. To show that $F(a) = \bigwedge^{\mathcal{B}}(F''A)$ we have to show that $[\phi] = \bigwedge^{\mathcal{B}}_{\psi \in X} [\psi]$. Clearly if $\psi \in X$, then $[\phi] \leqslant [\psi]$ because $\phi \overset{*}{\to} \psi$. Now suppose that $\chi \in T$ and $[\chi] \leqslant [\psi]$ for all $\psi \in X$. Then $\chi \overset{*}{\to} \psi$ for all $\psi \in X$, hence $\chi \overset{*}{\to} \bigwedge X$, but $\bigwedge X \overset{*}{\to} \phi$ so $\chi \overset{*}{\to} \phi$, i.e., $[\chi] \leqslant [\phi]$. Therefore $[\phi]$ is the greatest lower bound in $\mathcal{B}$ of $\{[\psi] \mid \psi \in X\}$. □

(Notice that the proof used 1.2 for formulas one of which is $\bigwedge X$, which need not be an element of T.)

We have embedded $\mathcal{B}_0$ completely in the countably generated B.a. $\mathcal{B}$, proving Kripke's embedding theorem.

Perhaps the proof seems based on a trick, but when one analyzes the general principles involved in it, one sees that no ingenuity is required once a few general simple facts are observed. The main point, besides ideas of § 1, may roughly be stated thus: In any Boolean model, the algebra of truth values has a complete embedding in an algebra of formulas, which takes every truth value to the sentences of which it is the value.

§ 5. Analysis of the proofs

The previous sections have shown the utility of certain general methods for the construction of B.a's. Here we want to formulate the construction of "algebras of formulas" (which goes back to Lindenbaum and Tarski) in a way which covers its previous as well as forthcoming applications in this work, and to reformulate the theorems of Gaifman–Hales and Kripke in a more detailed form which will be useful for several purposes. We also take the opportunity to hint at the constructive contents of the results by a suitable restriction of the methods of proof, but this will be studied elsewhere and here the reader is advised to observe only that the Axiom of Choice (AC) is not used and everything is done in ZF.

First let us describe the infinitary predicate-language in a more comprehensive way than in § 1. Call the symbols R_i^n (any $n < \omega$ and any i) *n-ary predicates*, and the symbols O_i^n, *n-ary operation symbols* (when $n = 0$ they give rise to constants). (The predicate $\prec$ or ϵ is identified with R_1^2 and P with R_0^1, say.) We also let u_i, v_i be a bindable and an unbindable variable for any i (not only an ordinal as in § 1), and for any a we have a name $\hat{a}$. These symbols are combined as usual to give terms and atomic formulas, and then 1.1 defines arbitrary formulas. In general we follow the syntactical notations and terminology of § 1–4. A language $\mathcal{L}$ is a set of relation – and operation – symbols. For a given language $\mathcal{L}$ and set A, those terms and formulas whose nonlogical symbols are in $\mathcal{L} \cup \{\hat{a} \mid a \in A\}$ are called $\mathcal{L}(A)$-terms (atomic-formulas) and $\mathcal{L}_{\infty\omega}(A)$-formulas, or simply $\mathcal{L}$-terms, $\mathcal{L}_{\infty\omega}$-formulas when $A = \emptyset$. When we talk of formulas, sentences etc., without qualifications, no restrictions on the non-logical symbols are assumed. The semantical notions connected with models and Boolean models for a language $\mathcal{L}$ are assumed known (cf. 3.2). Operation symbols are interpreted in Boolean models as in two-valued models. If M is a two-valued model and ϕ, ψ are $\mathcal{L}_{\infty\omega}(M)$-formulas, we say that ϕ implies ψ in M when for every (suitable) assignment f into M, $M \models \phi[f]$ implies $M \models \psi[f]$. The analogous notion for $\mathfrak{B}$-valued models defined in §4 (for any complete B.a. $\mathfrak{B}$; called there $\mathfrak{B}_1$). The relation "implies in M" (where M is any model or Boolean model for $\mathcal{L}$) satisfies 1.2 and 1.3 for all $\mathcal{L}_{\infty\omega}(M)$-formulas (and all variables $u = u_i$, $v = v_j$). Moreover, clauses (1), (3) of 1.3 can be strengthened to:

$$(\forall u)(\phi) \xrightarrow{*} \phi(\tfrac{u}{t}), \qquad \phi(\tfrac{u}{t}) \xrightarrow{*} (\exists v)(\phi),$$

where t is any $\mathcal{L}(M)$-term in which no bindable variable is free.

We shall now formulate the principle of construction of algebras of formula in a quite general way.

5.1. Theorem. Assumptions: (α) *Z is a set of unbindable variables.*

(β) *T is a non-empty set of formulas and:*

(1) *if $\phi, \psi \in T$, then $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi \in T$;*

(2) *if $\neg\psi \in T$ then $\psi \in T$; if $\bigwedge X \in T$ or $\bigvee X \in T$, then $X \subseteq T$; if $(\forall u)(\phi' \in T)$ or $(\exists u)\,\phi' \in T)$, then $\phi'(\tfrac{u}{v}) \in T$ for all $v \in Z$;*

(3) *if $\phi \in T$, then some $v \in Z$ is not free in ϕ.*

(γ) *$\xrightarrow{*}$ is a relation on T, and satisfies 1.2 and 1.3 for elements of T (i.e., when the formulas on both sides of $\xrightarrow{*}$ are assumed to be in T) with v in 1.3 restricted to Z.*

Conclusion: *There is a unique B.a. $\mathfrak{B}$ whose elements are the classes $[\phi]$ ($\phi \in T$) of an equivalence relation on T, such that for all $\phi, \psi \in T$ the following hold:*

(I) *$[\phi] \leqslant^{\mathfrak{B}} [\psi]$ iff $\phi \xrightarrow{*} \psi$, and $[\phi] = [\psi]$ iff $\phi \xrightarrow{*} \psi \xrightarrow{*} \phi$;*

(II) *if $\phi = \neg\psi$, then $[\phi] = \neg^{\mathfrak{B}}[\psi]$;*

(III) *if $\phi = \bigwedge X$, then ($X \subseteq T$ and) $[\phi] = \bigwedge_{\chi\in X}[\chi]$; dually for $\bigvee X$;*

(IV) *if $\phi = (\forall u)(\phi')$ then ($\phi'(\tfrac{u}{v}) \in T$ for all $v \in Z$ and) $[\phi] = \bigwedge_{v\in Z}[\phi'(\tfrac{u}{v})]$; dually for $(\exists u)(\phi')$.*

This B.a. $\mathfrak{B}$ is generated (in the $<\infty$ sense) by $\{[\phi] | \phi \in T, \phi$ atomic$\}$.

Proof: Left to the reader (see §1). □

5.2. Remark. (1) $\mathfrak{B}$ is uniquely determined by T and $\xrightarrow{*}$, so we shall sometimes denote it as $T/\xrightarrow{*}$. Conversely, T and $\xrightarrow{*}$ may be retrieved from $\mathfrak{B}$.

(2) Usually 5.1 will be used for $Z = \{v_n \mid n \in \omega\}$, as in § 1, §4. There a set T satisfying assumption (β) has been defined by means of cardinality conditions. We shall soon describe a more constructive and "economical" procedure for finding T (5.4).

(3) Let Z be a set of unbindable variables, X a set of atomic formulas. By an algebra of formulas over Z and X we mean a B.a. of the form $T/\overset{*}{\sim}$ where T and $\overset{*}{\sim}$ satisfy (β) and (γ) of 5.1 and $\{\phi \in T \mid \phi \text{ is atomic}) \subseteq X$

5.3. Definition. The set Sub(ϕ) of *subformulas of* ϕ is defined recursively by:

(1) $\text{Sub}(\phi) = \{\phi\}$ if ϕ is atomic;
(2) $\text{Sub}(\phi) = \{\phi\} \cup \text{Sub}(\psi)$ if $\phi = \neg\psi$;
(3) $\text{Sub}(\phi) = \{\phi\} \cup \bigcup_{\psi\in X} \text{Sub}(\psi)$ if $\bigwedge X$ or $\bigvee X$;
(4) $\text{Sub}(\phi) = \{\phi\} \cup \text{Sub}(\psi)$ if $\phi = (\forall u)(\psi)$ or $(\exists u)(\psi)$.

The set Sub*(ϕ) of subformulas in the Z-sense of ϕ, where Z is any set of terms, is defined in the same way, except that (4) is replaced by:

(4*) $\text{Sub}^*(\phi) = \{\phi\} \cup \bigcup_{t\in Z} \text{Sub}^*(\psi(^u_t))$ if $\phi = (\forall u)(\psi)$ or $(\exists u)(\psi)$.

In other words, $\psi \in \text{Sub}^*(\phi)$ iff there is a sequence $\langle \psi_k \mid k \leqslant n\rangle$ ($n \geqslant 0$) such that $\psi_0 = \phi$, $\psi_n = \psi$ and for each $k < n$, $\psi_k = \neg\psi_{k+1}$, or $\psi_k = \bigwedge X$ or $\bigvee X$ where $\psi_{k+1} \in X$, or $\psi_k = (\forall u)(\chi)$ or $(\exists u)(\chi)$ where $\psi_{k+1} = \chi(^u_t)$ for some $t \in Z$.

In 5.4 the notion of Sub* is used to construct sets T satisfying (β) of 5.1, and the reader should note that only the case $J = \{A \subseteq Z \mid A$ is finite$\}$ will be used here.

5.4. Theorem. *Let Z be an (infinite) set of unbindable variables, J a proper ideal of subsets of Z including all finite subsets ("proper" means $Z \notin J$). Let Y_0 be a non-empty set of formulas such that for all $\phi \in Y_0$, $\{v \in Z \mid v$ is free in $\phi\} \in J$. Put $Y = \bigcup_{\phi\in Y_0}\text{Sub}^*(\phi)$ (Sub* in the Z-sense), T = closure of Y under $\neg, \wedge, \vee$. Then 5.1(β) holds for T, and moreover $\{v \in Z \mid v$ is free in $\phi\} \in J$ for all $\phi \in T$.*

Proof (outlined). First show 5.1(β)(2) for Y instead of T, and deduce (β)(1), (β)(2) for T. Next show $\{v \in Z \mid v$ is free in $\phi\} \in J$ for all $\phi \in Y$, and extend this to all $\phi \in T$. Since $Z \notin J$, (β)(3) follows. □

Here is a strong version of the Gaifman–Hales theorem (in ZF).

5.5. Theorem. *Let Ω be an infinite set, and denote:*

$$Z = \{v_N \mid N \in \Omega\}, \qquad X_0 = \{v_M \in v_N \mid M, N \in \Omega\}.$$

For any set A an algebra $\mathcal{B} = T/\overset{*}{\to}$ of formulas over Z and X_0 can be found, such that for all $x \in A$, $N \in \Omega$, $\pi_x(v_N) \in T$ (see Definition 2.1 for π_x), and for each $N \in \Omega$, $\langle [\pi_x(v_N)] \mid x \in A\rangle$ is a family of non-zero pairwise-disjoint, hence distinct, elements of $\mathcal{B}$.

Thus $\mathcal{B}$ is generated by a set of the form $\{b_x \mid x \in \Omega \times \Omega\}$, and there exist 1–1 functions from A into $\mathcal{B}$.

Proof. We may assume that $A \neq \emptyset$. Let

$$Y_0 = \{\pi_x(v_N) \mid x \in A,\ N \in \Omega\}, \qquad Y = \bigcup_{\phi \in Y_0} \mathrm{Sub}^*(\phi)$$

(Sub* in the Z-sense) and T = closure of Y under $\neg, \wedge, \vee$. By 5.4 (with J the ideal of finite subsets of Z) T satisfies 5.1(β). 5.1(α) is obvious. Also, it is easy to see that for $\phi \in T$, $\{v \mid v \text{ is free in } \phi\} \in J$ (cf. the proof of 5.4), hence $\{\phi \in T \mid \phi \text{ is atomic}\} \subseteq X_0$. Put $D = \mathrm{TC}(A)$, and let $\phi \overset{*}{\to} \psi$ mean that ϕ implies ψ in the model $\langle D, \in_D\rangle$. Then 5.1(γ) holds and so $\mathcal{B} = T/\overset{*}{\to}$ is an algebra of formulas over Z and X_0, generated by $\{b_x \mid x \in \Omega \times \Omega\}$ where

$$b_{\langle M,N\rangle} = \begin{cases} [v_M \in v_N] & \text{if } (v_M \in v_N) \in T, \\ 0^{\mathcal{B}} & \text{otherwise.} \end{cases}$$

To complete the proof we need only show that for any $N \in \Omega$, $\langle [\pi_x(v_N)] \mid x \in A\rangle$ is a family of non-zero disjoint elements of $\mathcal{B}$. This is so because $\pi_x(v_N)$ is satisfiable in $\langle D, \in_D\rangle$ for $x \in A$, but $\pi_x(v_N) \wedge \pi_y(v_N)$ is not if $x \neq y$, as is clear from 2.2. □

5.6. Remark. When $\Omega = \omega$ we get the usual Gaifman–Hales theorem. But 5.5 says more because in ZF without AC, $\Omega \times \Omega$ need not have an infinite countable subset when Ω is infinite. It is interesting that all known proofs of the Gaifman–Hales theorem give generators indexed by pairs (at least), or use pairing functions on ω. I suspect that this use of pairs is essential, and one way of expressing this (not the only one) is by the following.

5.7. Conjecture. *The following statement is independent of ZF:*
For any infinite set Ω and any set A, there is a B.a. $\mathcal{B}$ generated by a set of the form $\{b_x \mid x \in \Omega\}$ and a 1–1 function $F : A \to \mathcal{B}$.
(*See note* (1) *on p.* 427.)

We now give the analogous form of Kripke's embedding theorem.

5.8. Theorem. *Let Ω be an infinite set, and denote*

$$Z = \{v_N \mid N \in \Omega\}, \qquad X_1 = \{v_M \in v_N \mid M, N \in \Omega\} \cup \{P(v_N) \mid N \in \Omega\}$$

For any B.a. $\mathcal{B}_0$ and function I_0 into $\mathcal{B}_0$, there is an algebra $\mathcal{B} = T/\overset{*}{\to}$ of formulas over Z and X_1 such that $\{\rho_x \mid x \in \mathrm{dom}(I_0)\} \subseteq T$ (see Theorem 3.4 for ρ_x), and a complete embedding F of $\mathcal{B}_0$ in $\mathcal{B}$ such that for each $x \in \mathrm{dom}(I_0)$, $F(I_0(x)) = [\rho_x]$.

Proof. Without loss of generality I_0 is onto $\mathcal{B}_0$. Let

$$Y_0 = \{\rho_x \mid x \in \mathrm{dom}(I_0)\}, \qquad Y = \bigcup_{\phi \in Y_0} \mathrm{Sub}^*(\phi)$$

and T = closure of Y under $\neg, \wedge, \vee$. Then 5.1(α), (β) hold and $\{\phi \in T \mid \phi \text{ atomic}\} \subseteq X_1$. Now proceed to define $\mathcal{B}_1, M, \overset{*}{\to}, \mathcal{B} = T/\overset{*}{\to}$ and F as in §4, choosing M (by 3.4) so that $\|\phi_x\|_M = I_0(x)$ for all $x \in \mathrm{dom}(I_0)$. Since I_0 is onto $\mathcal{B}_0$, (*) of §4 is satisfied, and the properties of $\mathcal{B}$ and F mentioned in §4 can be proved. The definition of F implies that $F(I_0(x)) = F(\|\rho_x\|_M) = [\rho_x]$ for $x \in \mathrm{dom}(I_0)$, which completes the proof. $\square$

5.9. Corollary. *Let Ω be an infinite set. Every B.a. has a complete embedding in some B.a. generated by a set of the form*

$$\{b_x \mid x \in \Omega \times \Omega\} \cup \{b'_x \mid x \in \Omega\}.$$

Next note that if Ω is infinite and Ω_1 obtained from it by omitting one element, then $\Omega \times \Omega$ has a subset which is the disjoint union of $\Omega_1 \times \Omega_1$ and a copy of Ω_1. Therefore the corollary entails the following.

5.10. Corollary. *Let Ω be an infinite set. Every B.a. $\mathcal{B}_0$ has a complete embedding in a B.a. $\mathcal{B}$ generated by a set of the form $\{b_x \mid x \in \Omega \times \Omega\}$.*

In contrast to the proof of 5.8 and 5.9, the proof of 5.10 depends on an arbitrary choice – the choice of an element from Ω (to get Ω_1). It is interesting to know whether (and in what sense) the choice is essential, but we shall not discuss this here.

There are several independence problems analogous to 5.7 which arise in relation to 5.10. For instance, is the statement obtained from 5.10 by replacing "$\Omega \times \Omega$" by "Ω" actually stronger than the one considered in 5.7 relative to ZF? Of course, both are theorems of ZFC.

From here onward we return to work in full ZFC.

§6. Towards quantitative forms of the theorems

The theorems formulated up to now have been "qualitative" in the sense that they asserted the existence of $(\aleph_0, <\infty)$-generated B.a's with various properties, without giving information about their cardinality or $(\aleph_0, <\kappa)$-generation. We now ask for more exact theorems, which will answer questions like: What is the maximum cardinality of an $(\aleph_0, <\kappa)$-generated B.a. (as a function of κ)? Which B.a's can be embedded in some $(\aleph_0, <\kappa)$-generated B.a.? Given a B.a. $\mathcal{B}$ what is the minimum power of an $\aleph_0$-generated B.a. $\mathcal{C}$ in which $\mathcal{B}$ can be (completely) embedded? What if $\mathcal{C}$ is required also to be complete? There are many other refinements and variations.

A beginning was made by Gaifman who showed [3, §6] that the maximum power of an $(\aleph_0, <\kappa)$-generated B.a. is at least κ (for regular κ), and at most a certain function of κ which coincides with κ under GCH (the generalized continuum hypothesis). We shall determine the exact value independently of GCH. This is made possible by the transfer of §2 from ordinals to arbitrary sets.

Our quantitative results are precise in the sense that they give full answers to the questions and not only upper and lower bound, so long as the questions deal with B.a's in general. When a completeness condition is imposed on the B.a's, the results are not so final, and different methods of attack are needed.

As a preparation we collect here some facts from cardinal arithmetics, leaving the proofs to the reader. Recall that $\nu, \nu_1, \ldots$ vary over *all* cardinals (see Preliminaries).

6.1. Notation. $\mathcal{P}(a) = \{x \mid x \subseteq a\}$, $\mathcal{P}_\nu(a) = \{x \mid x \subseteq a, |x| = \nu\}$, and $\mathcal{P}_{<\nu}(a)$, $\mathcal{P}_{\leqslant\nu}(a)$ are defined similarly. The weak power $\nu_1^{<\nu_2}$ is defined for $\nu_1 \geqslant 2$ by

$$\nu_1^{<\nu_2} = \sum_{\nu<\nu_2} \nu_1^\nu ,$$

and we find it convenient to let $0^{<\nu} = 1^{<\nu} = 2^{<\nu}$.

6.2. Fact. *If* $(\forall i \in I)\,(\nu_i > 0)$ *then*

$$\sum_{i\in I} \nu_i = \max(|I|, \sup_{i\in I} \nu_i) ,$$

except when I*, and each* ν_i*, are finite.*

6.3. Fact. (i) *Weak power is monotone in both arguments*;
(ii) $\nu \leqslant 2^{<\nu} \leqslant \nu_1^{<\nu}$ (*for all* ν_1*, including* 0, 1);
(iii) $\nu^{<\aleph_0} = \max(\nu, \aleph_0)$;
(iv) $\nu^{<\lambda'} = (\max(\nu, 2))^\lambda$;
(v) $\nu^{<\kappa} = \sup_{\lambda<\kappa} [(\max(\nu, 2))^\lambda]$; *if* κ *is a limit cardinal*;
(vi) *assuming* GCH, $2^{<\kappa} = \kappa$.

6.4. Fact. (i) *if* $|A| \geqslant \aleph_0, \nu$*, then* $|\mathcal{P}_\nu(A)| = |\mathcal{P}_{\leqslant\nu}(A)| = |A|^\nu$;
(ii) *if* $|A| \geqslant \aleph_0$, $|A|^+ \geqslant \nu$*, then* $|\mathcal{P}_{<\nu}(A)| = |A|^{<\nu}$.

6.5. Fact. (i) *if* $\nu < \kappa$*, then* $\nu^{<\kappa} = 2^{<\kappa}$, $\aleph_0^{<\kappa} = 2^{<\kappa}$;
(ii) *if* κ *regular and* $\nu \leqslant 2^{<\kappa}$*, then* $\nu^{<\kappa} = 2^{<\kappa}$;
(iii) *if* $\lambda \leqslant \mathrm{cf}(\kappa)$*, then* $(\nu^{<\kappa})^{<\lambda} = \nu^{<\kappa}$*, hence if* κ *is regular, then* $(\nu^{<\kappa})^{<\kappa} = \nu^{<\kappa}$.

As an illustration we prove the following:

6.6. Fact. *Let* $\mathcal{B}$ *be a B.a.,* $A \subseteq \mathcal{B}$*,* $C = [A]_{\mathcal{B}}^{<\kappa}$*. Then* $|C| \leqslant |A|^{<\kappa}$*. If* $|A| < \kappa$ *or* κ *is regular and* $|A| \leqslant 2^{<\kappa}$*, then* $|C| \leqslant 2^{<\kappa}$.

Proof. The second assertion follows from the first by 6.5(i)–(ii). We begin by proving $|C| \leqslant |A|^{<\kappa}$ for regular κ. Define $C_0 = A$ and for $\alpha > 0$, putting $C'_\alpha = \bigcup_{\beta<\alpha} C_\beta$, let

$$C_\alpha = C'_\alpha \cup \{\neg x \mid x \in C'_\alpha\} \cup \{\bigwedge X \mid X \in \mathcal{P}_{<\kappa}(C'_\alpha)\} \cup$$
$$\cup \{\bigvee X \mid X \in \mathcal{P}_{<\kappa}(C'_\alpha)\}$$

($\neg$, $\bigwedge$, $\bigvee$ are the operations of $\mathfrak{B}$, and for the moment let $\bigwedge X$ or $\bigvee X$ be 0 if the meet or join does not exists in $\mathfrak{B}$). It is clear, since κ is regular, that

$$C = \bigcup_\alpha C_\alpha = \bigcup_{\alpha<\kappa} C_\kappa .$$

Let $\mu = |A|^{<\kappa}$. By 6.5(iii) $\mu^{<\kappa} = \mu$. Hence we can prove by induction on $\alpha \leqslant \kappa$ that $|C_\alpha| \leqslant \mu$, using the facts that $|A| \leqslant \mu$, $\kappa \leqslant \mu$ and

$$\mu + \mu + |\mathcal{P}_{<\kappa}(\mu)| + |\mathcal{P}_{<\kappa}(\mu)| = \mu \cdot 4 = \mu .$$

Hence $|C| \leqslant \mu = |A|^{<\kappa}$. If κ is singular then we may write

$$C = \bigcup_{\lambda<\kappa} [A]^{<\lambda^+}_{\mathfrak{B}}$$

(see the definition of $[\ ]^{<\kappa}$ in the Preliminaries), hence

$$|C| \leqslant \sum_{\lambda<\kappa} |A|^{<\lambda^+} = \sup_{\lambda<\kappa} (\max|A|, 2)^\lambda = |A|^{<\kappa} .$$

A similar, even simpler, argument, proves the following:

6.7. Fact. $2^{<\kappa} = |\{x \subseteq \kappa \mid \operatorname{Sup} x < \kappa\}| = |H(\kappa)|$.

Recall that $H(\kappa) = \{x \mid |\mathrm{TC}(x)| < \kappa\}$.

§7. The cardinalities of countably generated B.a's

If $\mathfrak{B}$ is $(\nu, <\kappa)$-generated, then, by 6.6, $|\mathfrak{B}| \leqslant \nu^{<\kappa}$, hence $\mathfrak{B} \leqslant 2^{<\kappa}$ if $\nu \leqslant \aleph_0$ Since our problems are trivial for finite B.a's, we concentrate on $\nu = \aleph_0$ and ask about the cardinalities of $(\aleph_0, <\kappa)$-generated B.a's, and specifically whether the bound $2^{<\kappa}$ is attained. The following theorem gives a complete answer.

7.1. Theorem. *If $\lambda \leqslant 2^{<\kappa}$, then there is an $(\aleph_0, <\kappa)$-generated B.a. $\mathfrak{B}$ of power λ, containing λ disjoint elements. Hence, the possible powers of $(\aleph_0, <\kappa)$-generated B.a's are exactly all λ such that $\lambda \leqslant 2^{<\kappa}$.*

This will be proved by applying the results of §5.

Only the following lemma about algebras of formulas (in the sense of Remark 5.2(3)) has to be added to the "qualitative" results obtained there.

7.2. Lemma. *Let Z be a set of unbindable variables, X a set of atomic formulas, and $\mathfrak{B} = T/\overset{*}{\to}$ an algebra of formulas over Z and X. If $|X| \leqslant \nu$, $|Z| < \kappa$ and for all $\phi \in T$, $|\mathrm{Sub}(\phi)| < \kappa$ (see Definition 5.3), then $\mathfrak{B}$ is $(\leqslant\nu, <\kappa)$-generated (κ is any infinite cardinal).*

Proof. Let $B_0 = \{[\phi] \mid \phi \in T, \phi \text{ atomic}\}$. Since $|B_0| \leqslant |X| \leqslant \nu$, it suffices to show that B_0 generates $\mathfrak{B}$ in the $<\kappa$-sense. First assume κ regular. By induction on the depth $d(\phi)$ of $\phi \in T$, we shall show that $[\phi] \in [B_0]_{\mathfrak{B}}^{<\kappa}$. If ϕ is atomic or $\phi = \neg\psi$, there is no difficulty. If $\phi = \bigwedge A$ or $\phi = \bigvee A$ and (by the induction hypothesis) $[\psi] \in [B_0]_{\mathfrak{B}}^{<\kappa}$ for all $\psi \in A$, then $[\phi] \in [B_0]_{\mathfrak{B}}^{<\kappa}$ by 5.1(III), because $|A| < \kappa$ and κ is regular ($|A| < \kappa$ since $|\mathrm{Sub}(\phi)| < \kappa$). If $\phi = (\forall u)(\phi')$ or $\phi = (\exists u)(\phi')$ and $\phi'(\tbinom{u}{v}) \in [B_0]_{\mathfrak{B}}^{<\kappa}$ for all $v \in Z$, then $[\phi] \in [B_0]_{\mathfrak{B}}^{<\kappa}$ by 5.1(IV) since $|Z| < \kappa$ and κ is regular. Thus $\mathfrak{B} = [B_0]_{\mathfrak{B}}^{<\kappa}$ because $\mathfrak{B} = \{[\phi] \mid \phi \in T\}$.

If κ is singular then the argument above shows that for any regular $\mu > |Z|$,

$$[B_0]_{\mathfrak{B}}^{<\mu} \supseteq \{[\phi] \mid \phi \in T, |\mathrm{Sub}(\phi)| < \mu\}.$$

Since $\kappa = \sup\{\mu \mid |Z| < \mu < \kappa, \mu \text{ regular}\}$, we conclude that

$$[B_0]_{\mathfrak{B}}^{<\kappa} \supseteq \{[\phi] \mid \phi \in T, |\mathrm{Sub}(\phi)| < \kappa\} = \mathfrak{B}.$$

In any case, B_0 generates $\mathfrak{B}$ in the $<\kappa$-sense. □

Proof of Theorem 7.1. Without loss of generality, $\kappa = \min\{\kappa \mid \lambda \leqslant 2^{<\kappa}\}$, hence $\kappa \leqslant \lambda$. If $\kappa = \lambda = \aleph_0$ the B.a. of finite and cofinite subsets of Ω will do, so we may assume $\aleph_0 < \kappa \leqslant \lambda \leqslant 2^{<\kappa}$. Since $|H(\kappa)| = 2^{<\kappa}$, $H(\kappa)$ has a subset A of cardinality λ. We assert that the B.a. $\mathfrak{B} = T/\overset{*}{\to}$ constructed in the proof of Theorem 5.5 with $\Omega = \omega$ has the required properties. $\mathfrak{B}$ is an algebra of formulas over $\{v_n \mid n < \omega\}$ and $\{v_m \in v_n \mid m, n < \omega\}$, and $\langle [\pi_x(v_0)] \mid x \in A\rangle$ is a family of λ disjoint elements of $\mathfrak{B}$. Thus to see that $|\mathfrak{B}| = \lambda$ and $\mathfrak{B}$ is $(\aleph_0, <\kappa)$-generated it suffices (by 7.2) to prove that $|T| \leqslant \lambda$ and $|\mathrm{Sub}(\phi)| < \kappa$ for each $\phi \in T$.

Recall the definitions of Y_0, Y and T in the proof of 5.5. Suppose we have shown that for each $\phi \in Y_0$ there is a $\mu < \kappa$ such that all conjunctions and disjunctions in ϕ are of size $\leqslant \mu$. Then the same holds for each $\phi \in Y$, hence for each $\phi \in T$. But this implies that $|\mathrm{Sub}(\phi)| \leqslant \mu$ and $|\mathrm{Sub}^*(\phi)| \leqslant \mu$ (because Sub* is in the $\{v_n \mid n < \omega\}$-sense and $\mu \geqslant \aleph_0$). Thus $|\mathrm{Sub}(\phi)| < \kappa$ for all $\phi \in T$. Also

$$|Y| \leqslant \sum_{\phi \in Y_0} |\mathrm{Sub}^*(\phi)| \leqslant \lambda \cdot \kappa = \lambda$$

since $|Y_0| = |\{\pi_x(v_n) \mid x \in A, n < \omega\}| \leqslant |A| \cdot \omega = \lambda$, and it follows that $|T| \leqslant \lambda$. Thus the proof will be complete if we show that for each $x \in A$, $n < \omega$ there is a $\mu < \kappa$ such that in $\pi_x(v_n)$ occur only $\bigwedge$ and $\bigvee$ of size $\leqslant \mu$. But this is obvious because $A \subseteq H(\kappa)$ and so for each $x \in A$, $\mu = \max(\aleph_0, |\mathrm{TC}(x)|)$ is suitable. □

7.3. Corollary. *Every infinite cardinal is the power of some $\aleph_0$-generated B.a.*

§8. A quantitative form of Kripke's embedding theorem

Let $\mathfrak{B}_0$ be an infinite B.a., $|\mathfrak{B}_0| = \lambda$. In view of Kripke's theorem the following two questions naturally arise:

(1) What is the least power of some $\aleph_0$-generated B.a. $\mathfrak{B}$ in which $\mathfrak{B}_0$ can be (completely) embedded?

(2) What is the least κ such that $\mathfrak{B}_0$ has a (complete) embedding in some $(\aleph_0, <\kappa)$-generated B.a.?

Obviously the answer to (1) is some $\lambda_1 \geqslant \lambda$ and the answer to (2) is some $\kappa_1 \geqslant \min\{\kappa \mid \lambda \leqslant 2^{<\kappa}\}$ (because if $\mathfrak{B}$ is $(\aleph_0, <\kappa)$-generated, $|\mathfrak{B}| \leqslant 2^{<\kappa}$). This is so even if the embedding is not required to be complete. The following theorem shows that $\lambda_1 = \lambda$, $\kappa_1 = \min\{\kappa \mid \lambda \leqslant 2^{<\kappa}\}$, and thus the answers to (1) and (2) depend only on $|\mathfrak{B}_0|$, not on $\mathfrak{B}_0$ itself, and are the same whether the embedding is required to be complete or not.

8.1. Theorem. *Let $\mathfrak{B}_0$ be a B.a., $|\mathfrak{B}_0| = \lambda \leqslant 2^{<\kappa}$. Then there is an $(\aleph_0, <\kappa)$-generated B.a. $\mathfrak{B}$ such that $|\mathfrak{B}| = \lambda$ and $\mathfrak{B}_0$ has a complete embedding in $\mathfrak{B}$.*

Proof. Just as in the proof of 6.2, we may assume that $\kappa = \min\{\kappa \mid \lambda \leq 2^{<\kappa}\}$, and the case $\kappa = \aleph_0$ is trivial (take $\mathfrak{B} = \mathfrak{B}_0$), so that we are left with the situation $\aleph_0 < \kappa \leq \lambda \leq 2^{<\kappa}$.

Since $\lambda \leq 2^{<\kappa}$ we can choose some $A \subseteq H(\kappa)$ such that $|A| = \lambda = |\mathfrak{B}_0|$. Let I_0 be a function from A onto $\mathfrak{B}_0$.

We assert that the B.a. $\mathfrak{B} = T/\stackrel{*}{\leftrightarrow}$ constructed in the proof of 5.8 with $\Omega = \omega$ has the required properties. As shown there $\mathfrak{B}_0$ has a complete embedding in $\mathfrak{B}$, which is an algebra of formulas over $\{v_n \mid n < \omega\}$ and a countable set X_1. Thus, proving that $|\mathfrak{B}| = \lambda$ and $\mathfrak{B}$ is $(\aleph_0, <\kappa)$-generated reduces (as in the proof of 7.1) to showing that for each $\phi \in Y_0$ there is a $\mu < \kappa$ such that all Λ, V occurring in ϕ are of size $\leq \mu$. This time $Y_0 = \{\rho_x \mid x \in A\}$ and $A \subseteq H(\kappa)$, so the last assertion is true because if $x \in A$ then $\mu = \max(\aleph_0, |\mathrm{TC}(x)|)$ is good for ρ_x, and $\mu < \kappa$.

Note that 7.1 can be obtained by specialization from 8.1. Also the following special case ($\kappa = \aleph_1$) is worth noting:

Every B.a. of power $\leq 2^{\aleph_0}$ has a complete embedding in an $(\aleph_0, <\aleph_1)$-generated B.a.

§9. The cardinalities of complete countably-generated B.a's

Since the results of §7–8 depend essentially on incomplete B.a's ("algebras of formulas") it is natural to raise the following questions:

(1) What are the possible cardinalities of complete $\aleph_0$-generated (or $(\aleph_0, <\kappa)$-generated) complete B.a's?

(2) Given a B.a. $\mathfrak{B}_0$ how are the answers to the questions at the beginning of §8 affected by requiring the B.a. $\mathfrak{B}$ in which $\mathfrak{B}$ is embedded to be complete?

This section is devoted to the first question, and the next one to the second and to other remarks. Unfortunately, we do not know the full answers. What we shall prove concerning question (1) is this.

9.1. Theorem. (1) *If λ is the power of some complete $\aleph_0$-generated B.a., then there is a regular $\kappa > \aleph_0$ such that $\lambda = 2^{<\kappa}$ (that is, $\lambda = 2^{\mu}$ for some μ, or $\lambda = \sup_{\mu<\kappa} 2^{\mu}$ for some weakly inaccessible (= regular limit) cardinal κ).*

(2) *If λ is the power of some complete $(\aleph_0, <\mu)$-generated B.a., then there is a regular κ, $\aleph_0 < \kappa \leq \mu$, such that $\lambda = 2^{<\kappa}$.*

9.2. Theorem. *If κ is a successor cardinal ($\kappa = \kappa_1^+$), then there is an $(\aleph_0, <\kappa)$-generated B.a. of power $2^{<\kappa} = 2^{\kappa_1}$ (hence, if $\mu \geqslant \kappa$, this B.a. is $(\aleph_0, <\mu)$-generated).*

It will be seen that 9.2 is not a complete converse of 9.1, because it leaves open the existence of complete $\aleph_0$-generated B.a's of power $2^{<\kappa}$ when κ is weakly inaccessible. At the end of the section I shall formulate without proof stronger results, which leave the question open only when κ is very large.

Before proving 9.1 and 9.2 the following remark is in order: 9.2 is a direct application of our previous results. On the other hand 9.1 is proved by methods which have no connection with the other parts of this work, and is given here only for completeness. These methods are known from the literature, and 9.1 itself (or rather, 9.4) is very nearly approached in some places (e.g., specialize the results of [27] to $\aleph_0$-generated B.a's).

For a systematic treatment of the subject we need the following cardinal invariant of B.a's. For a B.a. $\mathfrak{B}$ let $CC(\mathfrak{B})$ (CC for "chain condition") be the smallest cardinal greater than the power of any set of pairwise-disjoint nonzero elements of $\mathfrak{B}$. If $CC(\mathfrak{B}) = \nu^+$ this means that ν is the maximum number of disjoint non-zero elements in $\mathfrak{B}$. It is easy to see that $CC(\mathfrak{B}) \neq \aleph_0$, but $CC(\mathfrak{B})$ may conceivably be a limit cardinal. We shall also use the well-known fact that if $A \subseteq \mathfrak{B}$, $a \in \mathfrak{B}$, $a = \mathrm{V}^{\mathfrak{B}} A$, then there is some $B \subseteq A$ such that $|B| < CC(\mathfrak{B})$ and $a = \mathrm{V}^{\mathfrak{B}} B$ (and dually). This implies that if $CC(\mathfrak{B}) \leqslant \kappa$, every $<\kappa$-subalgebra of $\mathfrak{B}$ is a $<\infty$-subalgebra, and every $<\kappa$-complete homomorphism from $\mathfrak{B}$ is $<\infty$-complete.

Let $\mathfrak{B}'$ be the normal completion of $\mathfrak{B}$. $\mathfrak{B}$ is dense in $\mathfrak{B}'$, hence $CC(\mathfrak{B}) = CC(\mathfrak{B}')$. If $A \subseteq \mathfrak{B}$, and A generates $\mathfrak{B}$ in the $<\infty$-sense, then A generates $\mathfrak{B}'$ in the $<\infty$-sense, and, by the previous paragraph, A generates $\mathfrak{B}$ and $\mathfrak{B}'$ in the $<\kappa$ sense provided $\kappa \geqslant CC(\mathfrak{B})$. As a last trivial observation we remark that if $\nu < CC(\mathfrak{B})$ and $\mathfrak{B}$ is complete then $|\mathfrak{B}| \geqslant 2^\nu$ since $\mathfrak{B}$ contains ν disjoint non-zero elements, and the join of any set of them.

We now quote a non-trivial theorem, which is a special case of [2, p. 220, Theorem 1].

9.3. Theorem (Erdös–Tarski). *If $\mathfrak{B}$ is an infinite B.a. then* CC($\mathfrak{B}$) *is a regular cardinal* ($>\aleph_0$).

9.4. Theorem. *If $\mathfrak{B}$ is a complete $\aleph_0$-generated B.a., $\kappa = \mathrm{CC}(\mathfrak{B})$, then* $|\mathfrak{B}| = 2^{<\kappa}$.

Proof. We already know (§6) that $|\mathfrak{B}| \leqslant 2^{<\kappa}$, since by the remarks above $\mathfrak{B}$ is $(\aleph_0, <\kappa)$-generated. On the other hand, if $\nu < \kappa$ then $|\mathfrak{B}| \geqslant 2^{\nu}$, so

$$|\mathfrak{B}| \geqslant \sup_{\nu<\kappa} 2^{\nu} = 2^{<\kappa}. \quad \square$$

Clearly Theorems 9.3 and 9.4 imply 9.1(1). To prove 9.1(2) assume that $\mathfrak{B}$ is complete and $(\aleph_0, <\mu)$-generated and let $\kappa^* = \mathrm{CC}(\mathfrak{B})$. Then $|\mathfrak{B}| = 2^{<\kappa^*}$ by 9.4. Since $|\mathfrak{B}| \leqslant 2^{<\mu}$ we get $2^{<\kappa^*} \leqslant 2^{<\mu}$. We cannot deduce that necessarily $\kappa^* \leqslant \mu$ (see below), but to prove 9 1(2) only the following lemma is needed.

9.5. Lemma. *If κ^* is regular and $2^{<\kappa^*} \leqslant 2^{<\mu}$, then there is a regular $\kappa \leqslant \mu$ such that $2^{<\kappa} = 2^{<\kappa^*}$.*

Proof. If $\kappa^* \leqslant \mu$ take $\kappa = \kappa^*$. So assume $\kappa^* > \mu$. Then $2^{<\kappa^*} \leqslant 2^{<\mu} \leqslant 2^{\mu} \leqslant 2^{<\kappa^*}$, hence $2^{<\kappa^*} = 2^{<\mu} = 2^{\mu}$. If μ is regular take $\kappa = \mu$. If μ is singular then it is a limit cardinal, and $2^{<\mu} = \sup_{\nu<\mu} 2^{\nu}$. We claim that $(\exists \nu < \mu)\,(2^{\nu} = 2^{<\mu})$. If not, then $2^{\nu} < 2^{<\mu} = 2^{\mu}$ for each $\nu < \mu$, and so, by König's lemma,

$$2^{\mu} = 2^{<\mu} = \sum_{\nu<\mu} 2^{\nu} < \prod_{\nu<\mu} 2^{\mu} \leqslant (2^{\mu})^{\mu} = 2^{\mu} ,$$

a contradiction. Thus there is a $\nu_0 < \mu$ such that $2^{\nu} = 2^{\mu}$ for all $\nu_0 \leqslant \nu < \mu$. μ is a limit cardinal, so we can take $\kappa = (\max(\aleph_0, \nu_0))^+$ and then κ is regular, $\kappa < \mu$, $2^{<\kappa} \geqslant 2^{\nu_0} = 2^{<\mu}$ hence $2^{<\kappa} = 2^{<\mu} = 2^{<\kappa^*}$. $\square$

This completes the proof of 9.1. The following question has risen in the course of this proof:

Let $\mathfrak{B}$ be a complete $(\aleph_0, <\mu)$-generated B.a. Does it follow that $\mathrm{CC}(\mathfrak{B}) \leqslant \mu$?

Under GCH the answer is affirmative, because if $\mathfrak{B}$ contains μ disjoint elements then $2^\mu \leqslant |\mathfrak{B}| \leqslant 2^{<\mu}$, i.e., $2^\mu = 2^{<\mu}$, while GCH implies $2^{<\mu} = \mu < 2^\mu$. However, using methods of [15] it can be shown that even for $\mu = \aleph_1$, the answer is independent of ZFC if ZFC is consistent. (See note (2) on page 427.)

Proof of Theorem 9.2. Let $\kappa = \kappa_1^+$. By Theorem 7.1 there is an $\aleph_0$-generated B.a. $\mathfrak{B}$ of power κ_1 such that $\mathfrak{B}$ contains κ_1 disjoint elements, hence $CC(\mathfrak{B}) = \kappa$. Let $\mathfrak{B}'$ be the normal completion of $\mathfrak{B}$. By the remarks preceding Theorem 9.3, $CC(\mathfrak{B}') = \kappa$, and hence $\mathfrak{B}'$ is $(\aleph_0, <\kappa)$-generated. By 9.4, $|\mathfrak{B}'| = 2^{<\kappa}$. Thus $\mathfrak{B}'$ is the desired B.a. □

In view of Theorem 9.4, the problem of determining the powers of complete, countably generated B.a's, reduces to that of finding which regular cardinals $\kappa > \aleph_0$ are of the form $CC(\mathfrak{B})$ for some countably-generated B.a. $\mathfrak{B}$. The proof of Theorem 9.2 shows that each successor cardinal is of this form. For readers acquainted with large cardinals, the following much stronger theorem, which has been proved using Solovay's method of almost disjoint sets, is given here without proof (see [25]).

9.6. Theorem. *If κ is regular, $>\aleph_0$ and not strongly-Mahlo, then $\kappa = CC(\mathfrak{B})$ for some complete $\aleph_0$-generated B.a. $\mathfrak{B}$.*

On the other hand, Menachem Magidor, and, I am informed, K.Kunen, have shown that if κ is weakly compact then there is no complete, $\aleph_0$-generated B.a. of power $2^{<\kappa} = \kappa$. The situation for cardinals κ which are strongly Mahlo (hence $2^{<\kappa} = \kappa$) and not weakly-compact is not clear. (See note (3) on page 427.)

We add one more remark. Pierce [16] has shown that λ is the power of some complete B.a. iff $\lambda = \lambda^{\aleph_0}$. It follows that there are cardinals λ which are powers of some complete B.a's but of no complete countably-generated B.a., because it is possible that $\lambda = \lambda^{\aleph_0}$, and $\lambda \neq 2^{<\kappa}$ for all regular κ (take any singular cardinal λ such that $cf(\lambda) > \aleph_0$ and $(\forall \mu < \lambda)(2^\mu < \lambda)$; then $\lambda = 2^{<\kappa}$ iff $\kappa = \lambda$, and $\lambda^{\aleph_0} = (2^{<\lambda})^{\aleph_0} = 2^{<\lambda} = \lambda$).

§10. Embedding in complete B.a's and other problems

10.1. Theorem. *Let $\mathfrak{B}_0$ be a B.a., $|\mathfrak{B}_0| = \lambda$. Then $\mathfrak{B}_0$ has a complete embedding in some complete $(\aleph_0, <\lambda^+)$-generated B.a. of power $\leqslant 2^\lambda$.*

Proof. By 8.1 $\mathfrak{B}_0$ has a complete embedding in an $(\aleph_0, <\lambda)$-generated B.a. $\mathfrak{B}$ of power λ. Let $\mathfrak{B}'$ be the normal completion of $\mathfrak{B}$. Then $|\mathfrak{B}'| \leqslant 2^\lambda$ (every element of $\mathfrak{B}'$ is determined by a cut in $\mathfrak{B}$), and $CC(\mathfrak{B}') = CC(\mathfrak{B}) \leqslant \lambda^+$ so that $\mathfrak{B}'$ is generated in the $<\lambda^+$-sense by the $\aleph_0$-generators of $\mathfrak{B}$. Clearly $\mathfrak{B}_0$ has a complete embedding in $\mathfrak{B}'$, so the proof is complete. □

Is Theorem 10.1 best possible? Obviously not in the same sense that Theorem 8.1 is, because for some choices of $\mathfrak{B}_0$ there may be a complete embedding in a complete $\aleph_0$-generated B.a. $\mathcal{C}$ of power $<2^\lambda$ ($\lambda = |\mathfrak{B}_0|$). On the other hand, if $|\mathfrak{B}_0| = \lambda$ and $\mathfrak{B}_0$ contains λ disjoint elements, then 2^λ is the minimum possible power for $\mathcal{C}$.

What about the least κ (call it κ_0) such that $\mathfrak{B}_0$ has a complete embedding is some complete $(\aleph_0, <\kappa)$-generated B.a. $\mathcal{C}$? By Theorem 10.1 $\kappa_0 \leqslant \lambda^+$. On the other hand, if $\mathfrak{B}_0$ contains λ disjoint elements there $|\mathcal{C}| \geqslant 2^\lambda$ hence $2^{<\kappa_0} \geqslant 2^\lambda$ and so $\kappa_0 < \lambda^+ \Rightarrow 2^{<\lambda} = 2^\lambda$, i.e., $2^{<\lambda} \neq 2^\lambda \Rightarrow \kappa_0 = \lambda^+$. Thus, if $2^{<\lambda} \neq 2^\lambda$, where $\lambda = |\mathfrak{B}_0|$, and $\mathfrak{B}_0$ contains λ disjoint elements, 10.1 cannot be improved for this $\mathfrak{B}_0$. Can the assumption $2^{<\lambda} \neq 2^\lambda$ (which follows from GCH) be dropped here? It could be dropped if we know that a complete $(\aleph_0, <\lambda)$-generated B.a. cannot contain λ disjoint elements, because this would refute $\kappa_0 \leqslant \lambda$. We arrive again at the question which has arisen from the proof of Theorem 9.1.

We conclude with a list of problems (or rather, directions of research) which we have not touched because the methods of this work are not especially suited to them. The completeness of B.a's, which has figured so much in the previous section, is only one member of a long list of useful properties. We could equally well ask about the possible powers of countably-generated B.a's which are, say, $<\mu$-complete (see [0]), or homogeneous, or which satisfy some chain condition etc. The methods of this work may help to give examples and positive results (like Theorem 9.2), but necessary conditions like 9.1, 9.4 seem to require combinatorial methods like those used by Pierce [16] or Erdös and Tarski [2].

In connection with Kripke's theorem, the following question presents itself: Can 10.1 be improved when $\mathfrak{B}_0$ is assumed to be complete? One might guess that then $\mathfrak{B}_0$ can be completely embedded in some complete countably generated B.a. of the same power. But this is not always the case, as the concluding remark of §9 shows.

One can also try to get Kripke-type theorems in which the initial B.a. $\mathcal{B}_0$ is assumed to have some other properties, and is embedded in an $\aleph_0$-generated B.a. with the same properties. The following theorem of [15] is a good example.

10.2. Theorem (Martin–Solovay). *If* $|\mathcal{B}_0| \leqslant 2^{\aleph_0}$ *and* $\mathrm{CC}(\mathcal{B}_0) \leqslant \aleph_1$, *then* $\mathcal{B}_0$ *has a complete embedding in a complete* $\aleph_0$*-generated B.a.* $\mathcal{B}$ *such that* $|\mathrm{CC}(\mathcal{B})| \leqslant \aleph_1$. *(The condition* $\mathrm{CC}(-) \leqslant \aleph_1$ *is called the countable chain condition.)*

The proof is based on the method of almost disjoint sets, which may lead to further theorems of this kind. The methods of the present work also lead to a variety of Kripke-type theorems for equational classes of B.a's and lattices, and we shall deal with them in Chapter II.

§11. Boolean terms, free B.a's and derivation systems

To construct Boolean terms (B.t's, cf. [3]) we start from symbols called atomic B.t's, which are to be regarded as variables (or, better, indeterminates or parameters) capable of assuming values in any B.a. To have an unlimited supply of atoms we take them as p_i for all i in the universe. B.t's are the atomic B.t's and everything obtainable from them by application of the symbolic operations $\neg$, $\wedge$, $\vee$ (as in the quantifier-free part of the predicate language). The idea is that once values have been assigned to some atomic B.t's in a B.a. $\mathcal{B}$, more complex B.t's can be evaluated by interpreting $\neg$, $\wedge$, $\vee$ as $\neg^{\mathcal{B}}$, $\wedge^{\mathcal{B}}$, $\vee^{\mathcal{B}}$ (at least if $\mathcal{B}$ is complete). Thus B.t's describe how elements of any (complete) B.a. may be generated from some given elements by the (infinitary) operations of that B.a., like polynomials on many indeterminates in algebra. Since such a description is useful for all B.a's, we do not assume completeness of $\mathcal{B}$ in the following definitions, which make precise the above idea of evaluating B.t's.

A valuation is a pair $(\mathcal{B}, I)$ in which $\mathcal{B}$ is a B.a. and I a function into $\mathcal{B}$. Think of I as assigning the value $I(x)$ to atomic B.t. p_x, for all $x \in \mathrm{dom}(I)$. Our aim is to define the value $\|\phi\| = \|\phi\|_{\mathcal{B},I}$ of the B.t. ϕ in the valuation $(\mathcal{B}, I)$. To take care of the fact that not every B.t. has a value in $\mathcal{B}$ under I (because it may contain occurrences of p_x for some $x \notin \mathrm{dom}(I)$, and because some meets and joins may fail to exist in $\mathcal{B}$), we choose an element $* \notin \mathcal{B}$ (say, $\mathcal{B}$ itself) as an "improper value" and give the definition recursively as follows:

$\|\phi\| = I(x)$ if $\phi = p_x$, $x \in \mathrm{dom}(I)$;

$\|\phi\| = \neg^{\mathcal{B}} \|\psi\|$ if $\phi = \neg\psi$, $\|\psi\| \in \mathcal{B}$;

$\|\phi\| = \bigwedge^{\mathcal{B}} A$ if $\phi = \bigwedge X$, $A = \{\|\psi\| \mid \psi \in X\}$, $A \subseteq \mathcal{B}$ and A has a meet in $\mathcal{B}$; dually for $\bigvee X$;

$\|\phi\| = *$ in any other case.

When $\|\phi\|_{\mathcal{B},I} \in \mathcal{B}$ we say that ϕ is *defined* in $(\mathcal{B}, I)$.

We remind the reader of some syntactical notations whose definitions for B.t's are just as for predicate formulas, ignoring the case of quantifiers. There are the usual notations $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \to \psi$, $\phi \leftrightarrow \psi$. $\mathrm{Sub}(\phi)$ is the set of subterms of ϕ (cf. Definition 5.3); $d(\phi)$ is the ordinal which measures the depth of ϕ (cf. Definition 1.4). Denote by $\mathrm{Atom}(\phi)$ the set of atomic B.t's occurring in ϕ. A B.t. ϕ is called finite when $\mathrm{Sub}(\phi)$ is a finite set. In general, we refer to $|\mathrm{Sub}(\phi)|$ as the length (or size) of ϕ. This turns out to be very natural for infinite B.t's, though somewhat arbitrary for finite B.t's.

As a first illustration of the usefulness of B.t's we give the following simple result.

11.1. Lemma. *Let* $(\mathcal{B}, I)$ *be a valuation,* $A = \mathrm{range}(I)$, κ *an infinite cardinal or* ∞. *Then*

$$[A]^{<\kappa}_{\mathcal{B}} = \{\|\phi\|_{\mathcal{B},I} \mid \phi \text{ is defined in } (\mathcal{B}, I) \text{ and is of length } <\kappa\}$$
$$[= \{\|\phi\|_{\mathcal{B},I} \mid |\mathrm{Sub}(\phi)| < \kappa\} \cap \mathcal{B}].$$

See Preliminaries for the definition of $[A]^{<\kappa}_{\mathcal{B}}$.

Proof. For regular κ, note that B.t's of length $<\kappa$ are closed under $\bigwedge$ and $\bigvee$ applied to sets of power $<\kappa$. Hence

$$C = \{\|\phi\|_{\mathcal{B},I} \mid \phi \text{ is defined in } (\mathcal{B}, I), |\mathrm{Sub}(\phi)| < \kappa\}$$

is easily seen to be the smallest $<\kappa$-subalgebra of $\mathcal{B}$ containing

$$\{\|p_x\|_{\mathcal{B},I} \mid x \in \mathrm{dom}(I)\} = \mathrm{range}(I) = A,$$

i.e. $C = [A]^{<\kappa}_{\mathfrak{B}}$. If κ is singular or ∞, then $[A]^{<\kappa}_{\mathfrak{B}} = \bigcup_{\lambda<\kappa} [A]^{<\lambda^+}_{\mathfrak{B}}$, and we go back to the regular case. □

This result explains our choice of the definition of $[A]^{<\kappa}_{\mathfrak{B}}$. It has been intended that generation in the $<\kappa$-sense will coincide with generation by means of B.t's of length $<\kappa$.

The main purpose of the introduction of B.t's in [3] has been the construction of free B.a's. Let us sketch now how this is done.

We shall say that a B.t. ϕ is supported by the set A when $\mathrm{Atom}(\phi) \subseteq \{p_x \mid x \in A\}$. For any B.t's ϕ, ψ let $\phi \overset{*}{\to} \psi$ mean that $\|\phi\| \leqslant \|\psi\|$ in every valuation in which ϕ and ψ are defined. If A is any set supporting both ϕ and ψ, one can restrict oneself to valuations $(\mathfrak{B}, I)$ where $\mathfrak{B}$ is complete and $\mathrm{dom}(I) = A$ (because passing to the normal completion of the B.a. does not affect the value of a B.t. in a valuation). The relation $\overset{*}{\to}$ clearly satisfies 1.2 for all B.t's.

Now put for any A and κ,

$$\mathrm{BT}^{<\kappa}(A) = \{\phi \mid \phi \text{ is supported by } A \text{ and its length is } <\kappa\}.$$

Since $\mathrm{BT}^{<\kappa}(A)$ is closed under $\neg, \wedge, \vee$ and subterms we may proceed as in §1 to divide it by the relation $\equiv$, where $\phi \equiv \psi$ iff $\phi \overset{*}{\to} \psi$ and $\psi \overset{*}{\to} \phi$, and get a B.a. $\mathfrak{B}$ of equivalence classes $[\phi]$ so that (I)–(III) of §1 hold for all $\phi, \psi \in \mathrm{BT}^{<\kappa}(A)$. Define $I : A \to \mathfrak{B}$ by $I(x) = [p_x]$ $(x \in A)$. Using (I)–(III) of §1 induction on ϕ shows that $\|\phi\|_{\mathfrak{B},I} = [\phi]$ for all $\phi \in \mathrm{BT}^{<\kappa}(A)$ Since $\mathfrak{B} = \{[\phi] \mid \phi \in \mathrm{BT}^{<\kappa}(A)\}$, Lemma 11.1 implies that $\mathfrak{B}$ is generated by range (I) in the $<\kappa$-sense.

The B.a. $\mathfrak{B}$ just defined is denoted by $\mathcal{F}^{<\kappa}(A)$. "$\mathcal{F}$" is for "free" and indeed we can give the following two characterizations of $(\mathfrak{B}, I)$ $(\mathfrak{B} = \mathcal{F}^{<\kappa}(A), I = \langle [p_x] \mid x \in A\rangle)$ which determine it up to isomorphism over A [i.e., if $(\mathfrak{B}_1, I_1)$, $(\mathfrak{B}_2, I_2)$ satisfy the conditions below, then $\mathrm{dom}(I_1) = \mathrm{dom}(I_2) = A$ and there is an isomorphism F of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ such that $I_2 = F \circ I_1$]. The first characterization is this: $I : A \to \mathfrak{B}$. $\mathfrak{B}$ is generated by range(I) is the $<\kappa$-sense, each $\phi \in \mathrm{BT}^{<\kappa}(A)$ is defined in $(\mathfrak{B}, I)$, and for all $\phi, \psi \in \mathrm{BT}^{<\kappa}(A)$, $\|\phi\|_{\mathfrak{B},I} = \|\psi\|_{\mathfrak{B},I}$ only if $\phi \equiv \psi$ (i.e., $\|\phi\| = \|\psi\|$ in every valuation in which both are defined).

The proof that this condition, which says that the generators obey no "constraints", characterizes $(\mathcal{F}^{<\kappa}(A), \langle [p_x] \mid x \in A\rangle)$ is trivial. Note also that if $|A| = |B|$, $\mathcal{F}^{<\kappa}(A) \cong \mathcal{F}^{<\kappa}(B)$.

Alternatively, we can characterize this valuation $(\mathfrak{B}, I)$, up to isomorphism as follows: $I : A \to \mathfrak{B}$, $\mathfrak{B}$ is generated by range(I) in the $<\kappa$-sense, and for every regular $\lambda \leqslant \kappa$ the following holds:

The B.a. $\mathfrak{B}_\lambda = [\mathrm{range}(I)]^{<\lambda}_{\mathfrak{B}}$ is $<\lambda$-complete and for every $<\lambda$-complete B.a. $\mathcal{C}$ and function $f : A \to \mathcal{C}$ there is a (unique) $<\lambda$-homomorphism $F: \mathfrak{B}_\lambda \to \mathcal{C}$ such that $F \circ I = f$.

The uniqueness of such a valuation $(\mathfrak{B}, I)$ up to isomorphism (when A is given) is easily seen, and so is the fact that $(\mathcal{F}^{<\kappa}(A), \langle [p_x] \mid x \in A \rangle)$ enjoys the above properties [when f is given, F is defined by $F([\phi]) = \|\phi\|_{\mathcal{C},f} (\phi \in \mathrm{BT}^{<\lambda}(A))$].

So far we have been following closely Gaifman's algebraic approach and terminology. There is an alternative approach to the subject which is presented in [10, chs. 4–6] (together with much additional material). In this metamathematical approach B.t's are identified with formulas of the infinitary propositional language and valuations with Boolean models for such a language (however Karp and Gaifman restrict themselves usually to valuations in complete B.a's, unlike the present work). The symbols $\neg$, $\bigwedge$, $\bigvee$ are then naturally called connectives. At times we shall find it helpful to take this view. For example, free B.a's are naturally arrived at as Lindenbaum algebras (= algebras of formulas) corresponding to Karp's "basic formal system", which is complete for validity in Boolean models. Karp goes even further to develop a representation theory for arbitrary B.a's [10, ch. 6].

However, we shall arrive at a derivation system equivalent to Karp's by a natural continuation of the algebraic approach followed thus far. Moreover, the system presented below, which is based on inequalities rather than single formulas (B.t's), generalizes easily from B.a's to lattices together with much of the material of this paper.

An inequality between B.t's is an expression $\phi \leqslant \psi$ where ϕ and ψ are B.t's. $(\phi \leqslant \psi)$ is meaningful (defined) in the valuation $(\mathfrak{B}, I)$ when ϕ and ψ are both defined in $(\mathfrak{B}, I)$, and then $(\phi \leqslant \psi)$ is true or false according as $\|\phi\|_{\mathfrak{B},I} \leqslant \|\psi\|_{\mathfrak{B},I}$ or not. Thus, an inequality may be viewed as the expression of a Boolean-algebraic relation which may hold among some elements of any B.a. (the values of the atomic B.t's occurring in the inequality). An equation can be represented by a pair of inequalities $\phi \leqslant \psi$, $\psi \leqslant \phi$ (alternatively inequalities can be eliminated in favor of equations; in B.a's, but not lattices, every inequality $(\phi \leqslant \psi)$ can be replaced by an equation $(\phi \to \psi) = 1$, so single B.t's can be used to express Boolean-algebraic relations.)

Let Γ be a set of inequalities, r an inequality. We say that r is a (Boolean) consequence of Γ when for every valuation $(\mathcal{B}, I)$ in which (every member of) Γ is true and r is defined, r is true. If all atoms occurring in Γ and r are from $\{p_x \mid x \in A\}$, then (using normal completions) it suffices to consider valuations $(\mathcal{B}, I)$ in which $\mathcal{B}$ is complete and $I : A \to \mathcal{B}$. The inequality r is (Boolean) valid when it is a consequence of $\emptyset$, i.e., true whenever defined.

The consequences of any set Γ of inequalities are easily seen to be closed under the following rules, which express the characterization of B.a's as complemented distributive lattices (cf. 1.2); we regard axiom schemes as rules without premises).

(1) $\quad \phi \leqslant \phi$;

(2) $\quad \dfrac{\phi \leqslant \chi, \chi \leqslant \psi}{\phi \leqslant \psi}$;

(3) $\quad \bigwedge X \leqslant \phi \quad (\phi \in X)$;

(4) $\quad \dfrac{\phi \leqslant \psi \quad (\text{all } \psi \in X)}{\phi \leqslant \bigwedge X}$;

(5), (6) – dual to (3), (4) (cf. 1.2);

(7) $\quad \phi \wedge (\chi \vee \psi) \leqslant \phi \wedge \chi \vee \phi \wedge \psi$;

(8) $\quad \phi \wedge \neg\phi \leqslant \psi ; \quad \psi \leqslant \phi \vee \neg\phi$.

Let us write $\Gamma \vdash r$ when the inequality r can be derived by rules (1)–(8), taking members of Γ as additional axioms. (A precise definition of "derivation from Γ" can be given in several well known ways and is left to the reader.) It is clear that if $\Gamma \vdash r$ then r is a Boolean consequence of Γ (in the proof it is convenient to consider valuations whose B.a's are complete, so that all B.t's occurring in the derivation are defined). To prove completeness of the system one proceeds as follows:

Let T be any non-empty set of B.t's closed under $\neg$, $\wedge$, $\vee$ and subterms and Γ any set of inequalities. Define the relation $\overset{*}{\to}$ on T by: $\phi \overset{*}{\to} \psi$ iff $\Gamma \vdash \phi \leqslant \psi$. Clearly 1.2 is satisfied for B.t's in T, hence a B.a. of equivalence classes can be formed so that (I)–(III) of §1 hold. This B.a. is denoted by $T/\Gamma \vdash$, so that, e.g., the $\mathcal{F}^{<\kappa}(A)$ previously defined is just $\mathrm{BT}^{<\kappa}(A)/\emptyset \vdash$. Let $\mathcal{B} = T/\Gamma \vdash$ and $I = \langle [p_x] \mid p_x \in T \rangle$. By induction on ϕ it is clear that $[\phi] = \|\phi\|_{\mathcal{B},I}$ for all $\phi \in T$. Hence, if $\phi, \psi \in T$ then $(\phi \leqslant \psi)$ is defined in $(\mathcal{B}, I)$, and it is true there if $[\phi] \leqslant [\psi]$ iff $\phi \overset{*}{\to} \psi$ iff $\Gamma \vdash \phi \leqslant \psi$.

It is useful to note also that if each $\phi \in T$ has length $<\kappa$ and $|\{x \mid p_x \in T\}| \leqslant \nu$, then $T/\Gamma\vdash$ is $(\leqslant\nu, <\kappa)$-generated.

Now to prove completeness of the above rules, suppose that r is a Boolean consequence of Γ, and choose T large enough so that the sides of all inequalities in $\Gamma \cup \{r\}$ belong to it. Let $\mathcal{B} = T/\Gamma\vdash$ and take I as above, so that for any $s \in \Gamma \cup \{r\}$, s is defined in $(\mathcal{B}, I)$ and true there iff $\Gamma \vdash s$. In particular, Γ is true in $(\mathcal{B}, I)$ and r is defined there, hence ture (being a consequence of Γ) hence $\Gamma \vdash r$. Thus:

$$(r \text{ is a consequence of } \Gamma) \quad \text{iff} \quad (\Gamma \vdash r)\,.$$

An analogous treatment of quantificational Boolean logic is rather straightforward. To avoid some trivial complications define an $\mathcal{L}_{\infty\omega}$-*inequality*, where $\mathcal{L}$ is any language (in the sense of §5), as an expression $\phi \leqslant \psi$, where ϕ and ψ are $\mathcal{L}_{\infty\omega}$-formulas in which no bindable variable is free. The $\mathcal{L}_{\infty\omega}$-inequality $\phi \leqslant \psi$ is said to hold in $(\mathcal{B}, M)$ (or just in M) where $\mathcal{B}$ is a complete B.a. and M a $\mathcal{B}$-valued model for $\mathcal{L}$, when ϕ implies ψ in M (i.e., $\|\phi[f]\| \leqslant \|\psi[f]\|$ for all assignments f into M). The class of $\mathcal{L}_{\infty\omega}$-inequalities that hold in M is closed under rules (1)–(8) above as well as (cf. 1.3):

(9) $(\forall u)(\phi) \leqslant \phi(^u_t)$, where t is an $\mathcal{L}$-term in which no bindable variable is free;

$$(10) \quad \frac{\phi \leqslant \psi(^u_v)}{\phi \leqslant (\forall u)(\psi)}\,,$$

where v is an unbindable variable not free in ϕ or in ψ;

(11), (12) – dual to (9), (10).

Usually we speak of (predicate-) inequalities without mentioning a language $\mathcal{L}$, that is – without restricting the predicates and operation-symbols which may occur in the inequality. Given a set Γ of inequalities and an inequality r, we say that r is a (Boolean) consequence of Γ when for every complete B.a. $\mathcal{B}$ and $\mathcal{B}$-valued model M whose language includes all non-logical sysmbols occurring in $\Gamma \cup \{r\}$, if (each member of) Γ holds in M, then r holds in M. Also let $\Gamma \vdash r$ mean that r can be derived from Γ by the rules (1)–(12) (the derivation consists of inequalities, and a limitation on their non-logical symbols is permissible but not necessary). Again we have, as in propositional logic:

$$r \text{ is a consequence of } \Gamma \quad \text{iff} \quad \Gamma \vdash r,$$

at least provided that Γ is closed ($(\phi \leq \psi) \in \Gamma$ implies that ϕ, ψ are sentences).

The proof is well-known but this is not the place to give it. It is based in the "only if" direction on the construction of a canonical (syntactical) model with truth-values in the normal completion of an algebra of formulas given by Theorem 5.1. A more refined use of the same idea leads to stronger completeness theorems as well as compactness theorems (cf. Karp's algebraic proof of Barwise's theorems concerning admissible sets). The individuals of the canonical model are terms of the language $\mathcal{L}$ for which it is a model. In the next sections we shall deal with the relation $\vdash$ in the predicate language only occasionally, and usually leave proofs to the reader. This material will not be needed for the Boolean algebraic results which we intend to prove. On the other hand, the relation $\vdash$ in the propositional language and its properties will prove quite useful, though strictly speaking they could be dispensed with. We shall write $\vdash r$ for $\emptyset \vdash r$, so that $\vdash r$ iff r is a valid inequality.

§12. A translation of predicate-formulas to B.t's and a syntactical counterpart of Kripke's embedding theorem

This section continues the work of §5 and is, like §5 independent of AC. Our first object is to specify the B.t's which describe how the "algebras of formulas" considered there are generated by the atomic formulas. This is done with the help of a translation QE (Quantifiers Elimination) from formulas (in the general sense of §5) to B.t's, based on the well-known idea of replacing $\forall$, $\exists$ by conjunction and disjunction over a fixed set of terms.

12.1. Definition. Let Z be a set of terms. The B.t. QE(ϕ, Z) (QE(ϕ) for short) is defined by recursion on the formula ϕ as follows:

$$\mathrm{QE}(\phi) = p_\phi \text{ if } \phi \text{ is atomic}$$

(this is somewhat arbitrary, we simply need distinct atomic B.t's to render distinct atomic formulas):

$$\mathrm{QE} \text{ commutes with } \neg, \wedge, \vee;$$

$$\mathrm{QE}((\forall u)(\psi)) = \bigwedge_{t\in Z} \mathrm{QE}(\psi(\tfrac{u}{t})), \text{ dually for } (\exists u)(\psi).$$

This translation has various applications, but the one needed here is the following.

12.2. Theorem. Assumptions: *as in Theorem* 5.1.
Conclusions: *as in Theorem* 5.1. *Moreover:*
Let $I = \langle [\phi] \mid \phi \in T,\ \phi$ *atomic*$\rangle$. *Then for each* $\phi \in T$,

$$[\phi] = \|\mathrm{QE}(\phi, Z)\|_{\mathcal{B},I}\ .$$

Proof. By induction on the depth $d(\phi)$ of $\phi \in T$. If ϕ is atomic then (writing $\|\cdot\|$ for $\|\cdot\|_{\mathcal{B},I}$)

$$[\phi] = I(\phi) = \|p_\phi\| = \|\mathrm{QE}(\phi, Z)\|\ .$$

The cases $\phi = \neg\psi$, $\phi = \wedge X$, $\vee\phi = X$ are based on (II), (III) of 5.1 and left to the reader. We shall only treat the case $\phi = (\forall u)(\psi)$ as an example. If $\phi = (\forall u)(\psi)$ then, by the induction hypothesis

$$[\psi(\tfrac{u}{v})] = \|\mathrm{QE}(\psi(\tfrac{u}{v}), Z)\| \quad \text{for all } v \in Z.$$

Denote the common value by b_v. Then, by 5.1(IV),

$$[\phi] = \bigwedge_{v\in Z}^{\mathcal{B}} b_v = \|\bigwedge_{v\in Z} \mathrm{QE}(\psi(\tfrac{u}{v}), Z)\| = \|\mathrm{QE}(\phi, Z)\|\ . \ \square$$

This result enables us to give a formulation of Theorems 5.5 and 5.8 in terms of B.t's. It is convenient to define the B.t's involved directly, without going back to predicate formulas.

Let Ω be an infinite set and q_{MN}, r_N $(M, N \in \Omega)$ distinct atomic B.t's fixed for the rest of this section.

12.3. Definition. The family $\langle \pi_x^N \mid N \in \Omega\rangle$ of B.t's is defined by $\in$-recursion on x as follows:

$$\pi_x^N = \bigwedge_M \left(q_{MN} \to \bigvee_{y \in x} \pi_y^M \right) \wedge \bigwedge_{y \in x} \bigvee_M (q_{MN} \wedge \pi_y^M) .$$

Let also $\rho_x^\circ = \bigvee_N (\pi_x^N \wedge r_N)$ for all x. (M, N vary over Ω.)

We leave it up to the reader to verify that in the special case $q_{MN} = p_{v_M \epsilon v_M}$, $r_N = p_{P(v_N)}$ one has: $\pi_x^N = \mathrm{QE}(\pi_x(v_N), Z)$ and $\rho_x^\circ = \mathrm{QE}(\rho_x, Z)$ where $Z = \{v_N \mid N \in \Omega\}$ and π_x, ρ_x are defined in 2.1, 3.4.

12.4. Theorem. (B.t's version of Theorem 5.5). *For any set A there is a valuation $(\mathfrak{B}, I)$ such that:*

(i) *for all x, $x \in \mathrm{dom}(I)$ iff $p_x = q_{MN}$ for some $M, N \in \Omega$;*

(ii) *each $b \in \mathfrak{B}$ has the form $\|\phi\|_{\mathfrak{B},I}$ for some B.t. ϕ, so that $\mathfrak{B}$ is generated by $\{\|q_{MN}\| \mid M, N \in \Omega\}$ ($\|\cdot\|$ is short for $\|\cdot\|_{\mathfrak{B},I}$);*

(iii) *for each $N \in \Omega$, $\langle \|\pi_x^N\| \mid x \in A \rangle$ is a family of non-zero pairwise-disjoint (hence distinct) elements of $\mathfrak{B}$.*

Proof. Without loss of generality $q_{MN} = p_{v_M \epsilon v_N}$. Let $\mathfrak{B} = T/\overset{*}{\to}$ be a B.a. as in 5.5 and let

$$I = \langle [\phi] \mid \phi \in T, \phi \text{ atomic} \rangle \cup$$
$$\cup \langle 0^{\mathfrak{B}} \mid \phi \in \{v_M \epsilon v_N \mid M, N \in \Omega\}, \phi \notin T \rangle .$$

Then $\mathrm{dom}(I) = \{v_M \epsilon v_N \mid M, N \in \Omega\}$ hence (i). To prove (ii) let $b \in \mathfrak{B}$. Then $b = [\phi]$ for some $\phi \in T$, hence by Theorem 12.2, $b = \|\mathrm{QE}(\phi, Z)\|$ where $Z = \{v_N \mid N \in \Omega\}$. (iii) follows from Theorem 5.5 because for $x \in A$, $N \in \Omega$ $[\pi_x(v_N)] = \|\mathrm{QE}(\pi_x(v_N))\| = \|\pi_x^N\|$ by Theorem 12.2.

12.5. Theorem (B.t's version of Theorem 5.8). *For any B.a. $\mathfrak{B}_0$ and function I_0 into it, there is a valuation $(\mathfrak{B}, I)$ such that:*

(i) *for all x, $x \in \mathrm{dom}(I)$ iff $p_x = q_{MN}$ or r_N for some $M, N \in \Omega$;*

(ii) *each $b \in \mathfrak{B}$ has the form $\|\phi\|_{\mathfrak{B},I}$ for some B.t. ϕ, so that $\mathfrak{B}$ is generated by*

$$\{\|q_{MN}\| \mid M, N \in \Omega\} \cup \{\|r_N\| \mid N \in \Omega\} ;$$

(iii) *there is a complete embedding F of $\mathfrak{B}_0$ in $\mathfrak{B}$ such that for each $x \in \mathrm{dom}(I_0)$, $F(I_0(x)) = \|\rho_x^0\|$. [ρ_x° has been defined in Definition 12.3.]*

Proof. Similar to that of Theorem 12.4, using 5.8 in place of 5.5. □

The reader will notice that 12.5 has the form of an "embedding theorem for valuations" $(\mathcal{B}_0, I_0) \to (\mathcal{B}, I)$, and the last equation of 5.7 may be rewritten as:

$$F(\|p_x\|_{\mathcal{B}_0, I_0}) = \|\rho_x^\circ\|_{\mathcal{B}, I}.$$

Since F is a complete embedding, one can "compute" $F(\|\phi\|_{\mathcal{B}_0 I_0})$ for every ϕ defined in $(\mathcal{B}_0, I_0)$. Let us proceed as follows:

Call a B.t. ϕ special when

$$\mathrm{Atom}(\phi) \subseteq \{q_{MN} \mid M, N \in \Omega\} \cup \{r_N \mid N \in \Omega\}\ .$$

ρ_x° is special for all x. Define a translation K of B.t's into special B.t's by

$$K(p_x) = \rho_x^0 \text{ (any } x\text{), and } K \text{ commutes with } \neg,\ \wedge,\ \vee.$$

Now we can state the following useful form of 12.5.

12.6. Theorem. *For each valuation* $(\mathcal{B}_0, I_0)$ *there is a valuation* $(\mathcal{B}, I)$ *such that* (i), (ii) *of* 12.5 *hold and also:*

(iii) *there is a complete embedding* F *of* $\mathcal{B}_0$ *in* $\mathcal{B}$ *such that for every B.t.* ϕ *defined in* $(\mathcal{B}_0, I_0)$,

$$F(\|\phi\|_{\mathcal{B}_0, I_0}) = \|K(\phi)\|_{\mathcal{B}, I}$$

where K *is the translation just defined.*

Proof of (iii). By induction on ϕ for the F of 12.5. □

Let us see what 12.4 and 12.6 mean from the point of view of the relation $\vdash$ defined in § 11 (recall that $\vdash r$ means $\emptyset \vdash r$, and also Definition 12.3).

12.7. Theorem. *For any* $N \in \Omega$ *and any* x, y *such that* $x \neq y$

$$\nvdash \pi_x^N \leqslant \pi_y^N\ .$$

(This gives a proper class of "incomparable" B.t's.)

Proof. By 12.4(iii) there is a valuation in which the inequality $\pi_x^N \leqslant \pi_y^N$ is false (take the set A of 12.4 so that $x, y \in A$), hence the assertion. □

Extend the translation K from B.t's to inequalities by $K(\phi \leqslant \psi) = (K(\phi) \leqslant K(\psi))$ and let $K(\Gamma) = \{K(s) \mid s \in \Gamma\}$ (Γ a set of inequalities).

12.8. Theorem. *For any set $\Gamma \cup \{r\}$ of inequalities, $\Gamma \vdash r$ iff $K(\Gamma) \vdash K(r)$*

Proof. ($\Rightarrow$). In a derivation of r from Γ replace each inequality s by $K(s)$. The result is a derivation of $K(r)$ from $K(\Gamma)$, since K preserves $\neg, \wedge, \vee$.

($\Leftarrow$). Suppose $\Gamma \nvdash r$ to prove $K(\Gamma) \nvdash K(r)$. r is not a consequence of Γ, hence there is a valuation $(\mathfrak{B}_0, I_0)$ in which Γ is true and r false. Let $(\mathfrak{B}, I), F$ be as in 12.6. Then by 12.6(iii), if s is an inequality defined in $(\mathfrak{B}_0, I_0)$ then $K(s)$ is defined in $(\mathfrak{B}, I)$ and s is true in $(\mathfrak{B}_0, I_0)$ iff $K(s)$ is true in $(\mathfrak{B}, I)$. In particular, $K(\Gamma)$ is true and $K(r)$ false in $(\mathfrak{A}, I)$. hence $K(\Gamma) \nvdash K(r)$.

12.9. Remark. Theorem 12.8 shows that a language with $\aleph_0$ propositional symbols is as rich as any propositional language in the sense that there is a $\neg, \wedge, \vee$-preserving translation of the latter into the former which is faithful w.r.t. the Boolean consequence relation ($\vdash$). In § 14 we shall derive from 12.8 an embedding theorem for free B.a's, and § 16 will make clear that 12.8 implies in fact Kripke's theorem and even 8.1.

We shall now sketch how K factors through a predicate-language, leaving proofs to the reader. Define the translation QI (Quantifiers Introduction) from B.t's to $\mathcal{L}_{\infty\omega}$-sentences ($\mathcal{L} = \{\epsilon, P\}$) by:

$$\mathrm{QI}(p_x) = \rho_x \text{ (any } x\text{) and QI commutes with } \neg, \wedge, \vee.$$

Let $\mathrm{QE}(\phi)$ (ϕ a predicate formula) be $\mathrm{QE}(\phi, \{v_N \mid N \in \Omega\})$. It is easy that if, for all $M, N \in \Omega$, $q_{MN} = \rho_{v_M \epsilon v_N}$, $r_N = \rho_{P(v_N)}$, then $K(\phi) = \mathrm{QE}(\mathrm{QI}(\phi))$ for each B.t. ϕ.

QI and QE are extended to inequalities and sets of inequalities in the obvious way.

12.10. Theorem. (I) *Let $\Gamma \cup \{r\}$ be a set of inequalities between B.t's. Then $\Gamma \vdash r$ iff $\mathrm{QI}(\Gamma) \vdash \mathrm{QI}(r)$.*

(II) *Let $\mathcal{L}$ be a set of predicates, Γ a set of closed $\mathcal{L}_{\infty\omega}$-inequalities (closed = with no free variables), r an $\mathcal{L}_{\infty\omega}$-inequality whose set of free* *variables is a finite subset of* $\{v_N \mid N \in \Omega\}$. *Then* $\Gamma \vdash r$ *iff* $\mathrm{QE}(\Gamma) \vdash \mathrm{QE}(r)$.

We shall not need 12.9, and the proof is not hard (for (II) the construction of canonical models is used as in the proof of completeness of the system of predicate logic presented in § 11; I have not tried to find weakest possible assumptions on $\mathcal{L}, \Gamma, r$ for the conclusion of (II) to hold).

§ 13. Cardinals related to $\mathcal{F}^{<\kappa}(\aleph_0)$

Our first aim is to show that the disjointness of π_x^N and π_y^N (see Definition 12.3) for $x \neq y$ is derivable. That is, for Ω, q_{MN}, r_N $(M, N \in \Omega)$ as in § 12, we claim:

13.1. Theorem. *If* $N \in \Omega$, $x \neq y$, *then* $\vdash \pi_x^N \wedge \pi_y^N \leqslant \bigvee \emptyset$.

Proof. We shall show that $(\forall N \in \Omega) \vdash \pi_x^N \wedge \pi_y^N \leqslant \bigvee \emptyset$ when $x \neq y$ by induction on x, y, the induction hypothesis being

$$(\forall u \in x)\,(\forall v \in y)\,[u \neq v \Rightarrow (\forall N \in \Omega) \vdash \pi_u^N \wedge \pi_v^N \leqslant \bigvee \emptyset]\,.$$

Since $x \neq y$ we may assume that $x \nsubseteq y$ and let $z \in x \sim y$. It follows from the definitions that for all $N \in \Omega$,

$$\vdash \pi_x^N \leqslant \bigwedge_{u \in x} \bigvee_M (q_{MN} \wedge \pi_u^M)\,, \qquad \vdash \pi_y^N \leqslant \bigwedge_M \left(q_{MN} \to \bigvee_{v \in y} \pi_v^M\right).$$

Since

$$\vdash \bigwedge_{u \in x} \bigvee_M (q_{MN} \wedge \pi_u^M) \leqslant \bigvee_M (q_{MN} \wedge \pi_z^M)\,,$$

it follows easily that

$$\vdash \pi_x^N \wedge \pi_y^N \leqslant \bigvee_M (q_{MN} \wedge \pi_z^M) \wedge \bigwedge_M \left(q_{MN} \to \bigvee_{v \in y} \pi_v^M\right).$$

Now, it is not hard to see that

$$\vdash \bigvee_M (q_{MN} \wedge \pi_z^M) \wedge \bigwedge_M \left(q_{MN} \to \bigvee_{v \in y} \pi_v^M\right) \leqslant \bigvee_M \bigvee_{v \in y} (\pi_z^M \wedge \pi_v^M)\,,$$

hence

$$\vdash \pi_x^N \wedge \pi_y^N \leqslant \bigvee_{\substack{M\in\Omega \\ v\in y}} (\pi_z^M \wedge \pi_v^M).$$

But $z \notin y$, hence by the induction hypothesis $\vdash \pi_z^M \wedge \pi_v^M \leqslant \mathrm{V}\emptyset$ for all $M \in \Omega$, $v \in y$, therefore $\vdash \pi_x^N \wedge \pi_y^N \leqslant \mathrm{V}\emptyset$. □

Note that Theorem 12.7 implies $\nvdash \pi_x^N \leqslant \emptyset$ (because if $\vdash \phi \leqslant \mathrm{V}\emptyset$ then $\vdash \phi \leqslant \psi$ for all ψ).

From now on we agree that $\Omega = \omega$ and $q_{mn} = p_{(m,n)}$, $r_n = p_n$ for $m, n < \omega$. This is permissible because $(\omega \times \omega) \cap \omega = \emptyset$ so the q_{mn}'s and r_n's do not coincide. Let λ be any infinite cardinal. If $|TC(x)| \leqslant \lambda$, then in π_x^n and ρ_x^0, as defined in 12.3, no conjunction or disjunction acts on more than λ B.t's, so the length of π_x^n and of ρ_x^0 is $\leqslant \lambda$. Thus $\pi_x^n \in \mathrm{BT}^{<\lambda^+}(\omega \times \omega)$ and $\rho_x^0 \in \mathrm{BT}^{<\lambda^+}(\omega \times \omega \cup \omega)$. If $\kappa > \aleph_0$ then $\kappa = \sup_{\lambda<\kappa} \lambda^+$, hence: if $x \in H(\kappa)$, then $\pi_x^n \in \mathrm{BT}^{<\kappa}(\omega \times \omega)$ and $\rho_x^0 \in \mathrm{BT}^{<\kappa}(\omega \times \omega \cup \omega)$. Recall that $\mathcal{F}^{<\kappa}(A) = \mathrm{BT}^{<\kappa}(A)/\emptyset\vdash$ (see §11). So for all $x \in H(\kappa)$, $n \in \omega$, the equivalence classes $[\pi_x^n]$ are members of $\mathcal{F}^{<\kappa}(\omega \times \omega)$. But since $\nvdash \pi_x^n \leqslant \mathrm{V}\emptyset$, these members are $\neq 0$, and 13.1 implies that for any fixed n they are pairwise disjoint. Since x varies on $H(\kappa)$, whose power is $2^{<\kappa}$, we obtain $2^{<\kappa}$ disjoint elements in $\mathcal{F}^{<\kappa}(\omega \times \omega)$. On the other hand $\mathcal{F}^{<\kappa}(\omega \times \omega)$ (is isomorphic to $\mathcal{F}^{<\kappa}(\aleph_0)$ and) is $(\aleph_0, <\kappa)$-generated and so its power is exactly $2^{<\kappa}$. The following theorem contains this conclusion but also supplements Theorem 7.1.

13.2. Theorem. *Let κ be an infinite cardinal.*

(1) $\mathcal{F}^{<\kappa}(\aleph_0)$ *has cardinality* $2^{<\kappa}$ *and contains* $2^{<\kappa}$ *disjoint elements;*

(2) *for any* $\lambda \leqslant 2^{<\kappa}$, $\mathcal{F}^{<\kappa}(\aleph_0)$ *has a subalgebra* $\mathcal{B}$ *which is* $(\aleph_0, <\kappa)$-*generated, has cardinality* λ *and contains* λ *disjoint elements.*

Proof. (1) has already been proved for $\kappa > \aleph_0$. Both (1), (2) are trivial for $\kappa = \aleph_0$, and it suffices to prove (2) when $\kappa = \min\{\kappa \mid \lambda \leqslant 2^{<\kappa}\}$. So all is reduced to the case $\aleph_0 < \kappa \leqslant \lambda \leqslant 2^{<\kappa}$, and $\mathcal{F}^{<\kappa}(\aleph_0)$ may be replaced by $\mathcal{F}^{<\kappa}(\omega \times \omega)$. Now we follow an already familiar procedure. Choose some $A \subseteq H(\kappa)$ such that $|A| = \lambda$, and let $Y_0 = \{\pi_x^n \mid x \in A, n \in \omega\}$, $Y = \bigcup_{\phi \in Y_0} \mathrm{Sub}(\phi)$ and $T =$ closure of Y under $\neg$, $\wedge$, $\vee$. As in proof of 7.1 (but more simply, for the distinction between Sub and Sub* does not appear) it is seen that $T \subseteq \mathrm{BT}^{<\kappa}(\omega \times \omega)$ and $|T| \leqslant \lambda$. Also T is closed under $\neg$, $\wedge$, $\vee$ and subterms. Thus the B.a. $T/\emptyset\vdash$ is $(\aleph_0, <\kappa)$-generated (by $\{[\phi] \mid \phi \in T, \phi \text{ atomic}\}$), has power $\geq \lambda$, and contains λ disjoint elements $[\pi_x^0]$, $x \in A$. It is obviously isomorphic to a subalgebra of $\mathcal{F}^{<\kappa}(\omega \times \omega)$, namely the subalgebra whose elements are $\{[\phi] \mid \phi \in T\}$ (note the ambiguous use of the symbol [ ] for equivalence classes within different sets, in this case T and $\mathrm{BT}^{<\kappa}(\omega \times \omega)$). This completes the proof of 13.1. □

13.3. Remark. The fact that the results of §6 and this section give exact cardinals, not only upper and lower bounds, is due to the transfer of §2 from ordinals to arbitrary sets. Otherwise we could only get $\kappa \leq |\mathcal{F}^{<\kappa}(\aleph_0)| \leq 2^{<\kappa}$, which follows already from the previously published proofs of the Gaifman–Hales theorem (cf. [3, §6]).

§14. Embedding in $\mathcal{F}^{<\kappa}(\aleph_0)$

We continue to use results of §12 with $\Omega = \omega$, $q_{mn} = p_{(m,n)}$, $r_n = p_n$. Then for B.t. ϕ, $K(\phi)$ (defined before Theorem 12.6) is a B.t. such that

$$\mathrm{Atom}(K(\phi)) \subseteq \{p_x \mid x \in (\omega \times \omega) \cup \omega\},$$

and by 12.8 $\vdash \phi \leq \psi$ iff $\vdash K(\phi) \leq K(\psi)$.

As remarked in §13, for all $\kappa > \aleph_0$ and all $x \in H(\kappa)$, $\rho_x^0 \in \mathrm{BT}^{<\kappa}(\omega \times \omega \cup \omega)$ But $\rho_x^0 = K(p_x)$, and $\mathrm{BT}^{<\kappa}(A)$ is closed (for any A) under $\neg$ and under and $\wedge$, $\vee$ applied to sets of power $< \mathrm{cf}(\kappa)$. Hence, by induction on ϕ, if $\phi \in \mathrm{BT}^{<\mathrm{cf}(\kappa)}(H(\kappa))$, then $K(\phi) \in \mathrm{BT}^{<\kappa}(\omega \times \omega \cup \omega)$, assuming $\kappa > \aleph_0$. This means that for any $A \subseteq H(\kappa)$, the equation $g([\phi]) = [K(\phi)]$ ($\phi \in \mathrm{BT}^{<\mathrm{cf}(\kappa)}(A)$) defines a 1–1 function from $\mathcal{F}^{<\mathrm{cf}(\kappa)}(A)$ $(= \mathrm{BT}^{<\mathrm{cf}(\kappa)}(A)/\emptyset\vdash)$ into $\mathcal{F}^{<\kappa}(\omega \times \omega \cup \omega)$. Noting that K preserves $\neg$, $\wedge$, $\vee$ and $\mathrm{BT}^{<\mathrm{cf}(\kappa)}(A)$ is closed under $\wedge$, $\vee$ applied to $<\mathrm{cf}(\kappa)$ B.t.'s, it is easily seen that g is a $<\mathrm{cf}(\kappa)$-complete embedding of $\mathcal{F}^{<\mathrm{cf}(\kappa)}(A)$ in $\mathcal{F}^{<\kappa}(\omega \times \omega \cup \omega)$. The power of A may be any $\nu \leq |H(\kappa)| = 2^{<\kappa}$, hence:

14.1. Theorem. *Let κ be an infinite cardinal, ν any cardinal, $\nu \leq 2^{<\kappa}$. Then there is a $<\mathrm{cf}(\kappa)$-complete embedding of $\mathcal{F}^{<\mathrm{cf}(\kappa)}(\nu)$ in $\mathcal{F}^{<\kappa}(\aleph_0)$. In particular, if κ is regular, $\nu \leq 2^{<\kappa}$, then the free $<\kappa$-complete B.a. on ν generators has a $<\kappa$-complete embedding in the free $<\kappa$-complete B.a. on $\aleph_0$-generators.*

Actually we have proved 14.1 only for $\kappa > \aleph_0$, but it is trivially true for $\kappa = \aleph_0$.

14.2. Corollary. *For any cardinal ν and infinite cardinal μ, $\mathcal{F}^{<\mu}(\nu)$ has a $<\mathrm{cf}(\mu)$-complete embedding in $\mathcal{F}^{<\kappa}(\aleph_0)$ for all large enough κ, in fact for all κ such that $\mu \leqslant \mathrm{cf}(\kappa)$ and $\nu \leqslant 2^{<\kappa}$.*

Proof. Let κ satisfy these two inequalities. The trivial embedding $\mathcal{F}^{<\mu}(\nu) \to \mathcal{F}^{<\mathrm{cf}(\kappa)}(\nu)$ is $<\mathrm{cf}(\mu)$-complete (for $\mathrm{BT}^{<\mu}(\nu) \subseteq \mathrm{BT}^{<\mathrm{cf}(\kappa)}(\nu)$ and $\mathrm{BT}^{<\mu}(\nu)$ is closed under $\bigwedge$, $\bigvee$ of $<\mathrm{cf}(\mu)$ B.t's). By 14.1 there is an embedding $\mathcal{F}^{<\mathrm{cf}(\kappa)}(\nu) \to \mathcal{F}^{<\kappa}(\aleph_0)$ which is $<\mathrm{cf}(\kappa)$-complete. Since $\mathrm{cf}(\mu) \leqslant \mu \leqslant \mathrm{cf}(\kappa)$, the composition

$$\mathcal{F}^{<\mu}(\nu) \to \mathcal{F}^{<\mathrm{cf}(\kappa)}(\nu) \to \mathcal{F}^{<\kappa}(\aleph_0)$$

gives a $<\mathrm{cf}(\mu)$-complete embedding (See note (4) on p. 427.) □

14.3. Remark. I do not know if 14.1 and 14.2 are best possible. Their interest lies in the fact that they give each free B.a. as a subalgebra of a free B.a. on $\aleph_0$-generators. Actually we can say more. Theorem 12.8 has been used here only for $\Gamma = \emptyset$, so the results are about B.a's of the form $T/\emptyset\vdash$. By using it for arbitrary Γ we can get embeddings of the form $(T/\Gamma\vdash) \to T'/K(\Gamma)\vdash$ for a suitable T'. Thus if $\mathfrak{B}_0$ is "nearly free" (i.e., isomorphic to some $T/\Gamma\vdash$ where Γ is small or simple in some sense) then it has a rather complete embedding in a nearly free B.a. on $\aleph_0$-generators. It remains to be seen whether such results can be embedded in a coherent theory and deserve a closer study.

§15. Another class of independent sentences

The sentences we are going to define are simpler than the ρ_x's of §3, and they will be used to extend previous results from countably-generated to $\leqslant\aleph_\alpha$-generated B.a's (any α). The idea is that allowing names for elements of $\aleph_\alpha$ one can get, when $\kappa \leqslant \aleph_\alpha$, $\aleph_\alpha^{<\kappa}$-independent sentences each of length $<\kappa$ through locating formulas for elements of $\mathcal{P}_{<\kappa}(\aleph_\alpha)$.

We shall work in the language $\mathcal{L} = \{\approx, \epsilon, P\}$ ($\approx$ is the equality predicate), and use also names $\hat{y}$ for arbitrary sets y. As a locating formula for the set x take:

$$\bar{\pi}_x(v_0) = (\forall u_0)\left(u_0 \in v_0 \leftrightarrow \bigvee_{y \in x} (u_0 \approx \hat{y})\right),$$

and then define $\bar{\rho}_x = (\exists u_1)(\bar{\pi}_x(u_1) \wedge P(u_1))$.

Next define a translation $\overline{\text{QI}}$ (cf. end of § 12) from B.t's to predicate-sentences by: $\overline{\text{QI}}(p_x) = \bar{\rho}_x$ (any x) and $\overline{\text{QI}}$ commutes with $\neg$, $\wedge$, $\bigvee$.

The independence of the sentences $\bar{\rho}_x$ is expressed in the following analogue of Theorem 3.2.

15.1. Theorem. *Let $\mathfrak{B}$ be a complete B.a. and I a function into $\mathfrak{B}$. Then there is a $\mathfrak{B}$-valued model M for $\mathcal{L}$ such that for all $x \in \text{dom}(I)$, $\bar{\rho}_x$ is an $\mathcal{L}_{\infty\omega}(M)$-sentence and $\|\bar{\rho}_x\|_M = I(x)$, and, more generally, if ϕ is a B.t., $\text{Atom}(\phi) \subseteq \{p_x \mid x \in \text{dom}(I)\}$, then $\|\phi\|_{\mathfrak{B},I} = \|\overline{\text{QI}}(\phi)\|_M$.*

Proof. Let M be a transitive non-empty set $\supseteq \text{dom}(I)$, in which $\approx$, ϵ, P are interpreted so that for all $x, y \in M$

$$\|\hat{x} \approx \hat{y}\|_M = \begin{cases} 1^{\mathfrak{B}} & \text{if } x = y, \\ 0^{\mathfrak{B}} & \text{otherwise;} \end{cases}$$

$$\|\hat{x} \in \hat{y}\|_M = \begin{cases} 1^{\mathfrak{B}} & \text{if } x \in y, \\ 0^{\mathfrak{B}} & \text{otherwise;} \end{cases}$$

$$\|P(\hat{x})\|_M = \begin{cases} I(x) & \text{if } x \in \text{dom}(I), \\ 0^{\mathfrak{B}} & \text{otherwise;} \end{cases}$$

It is easy to see, using the transitivity of M, that if $x, a \in M$ then

$$\|\bar{\pi}_x(\hat{a})\|_M = \begin{cases} 1^{\mathfrak{B}} & \text{if } x = a, \\ 0^{\mathfrak{B}} & \text{otherwise,} \end{cases}$$

and to deduce that if $x \in \text{dom}(I)$, $\|\bar{\rho}_x\|_M = I(x)$. The last assertion of 15.1 is now proved by induction on ϕ. □

Now let Ω be an infinite set and denote $Z = \{v_N \mid N \in \Omega\}$. For any set D put

$$X_D = \{v_M \in v_N \mid M, N \in \Omega\} \cup \{P(v_N) \mid N \in \Omega\} \cup$$
$$\cup \{v_M \approx \hat{y} \mid M \in \Omega, y \in D\}.$$

The analogue of Theorem 5.8 runs as follows (the proof is left to the reader):

15.2. Theorem. *For any valuation* $(\mathfrak{B}_0, I_0)$ *there are a set* D, *an algebra* $\mathfrak{B} = T/\stackrel{*}{\to}$ *of formulas over* Z *and* X_D *such that* $\{\bar{\rho}_x \mid x \in \mathrm{dom}(I_0)\} \subseteq T$ *and a complete embedding* F *of* $\mathfrak{B}_0$ *in* $\mathfrak{B}$ *such that for each* $x \in \mathrm{dom}(I_0)$, $F(I_0(x)) = [\rho_x]$. (*In case* $\mathrm{range}(I_0) = \mathfrak{B}_0$ *take* $D = \bigcup \mathrm{dom}(I_0)$.)

From this result an analogue of Theorem 8.1 could be derived, but we prefer to postpone the discussion of cardinals (and the use of AC) to §16, and give here the analogues of 12.5, 12.8.

Let Ω and Z be as above, and let QE$(\cdot)$ be short for QE$(\cdot, Z)$ as defined in 12.1. It is natural to consider the following translation from B.t's to B.t's:

$$\bar{K}(\phi) = \mathrm{QE}(\mathrm{QI}(\phi)).$$

Writing QE$(\bar{\rho}_x)$ in full we obtain the following direct definition of $\bar{K}$:

15.3. Definition. $\bar{K}(\rho_x) = \bigvee_N [\bigwedge_M (q_{MN} \leftrightarrow \bigvee_{y \in x} r_{My}) \wedge s_N]$ and $\bar{K}$ commutes with $\neg$, $\wedge$, $\vee$.

Here M, N vary on Ω, and $q_{MN} = p_{v_M \in v_N}$, $r_{My} = p_{v_M \approx \hat{y}}$, $s_N = p_{P(v_N)}$, but of course it is only essential that q_{MN}, r_{My}, s_N ($M, N \in \Omega$ and any y) are distinct B.t's. As in the proof of 12.5, 12.6, one concludes from 15.2 the following:

15.4. Theorem. *For each valuation* $(\mathfrak{B}_0, I_0)$ *there is a valuation* $(\mathfrak{B}, I)$ *and a complete embedding* F *of* $\mathfrak{B}_0$ *in* $\mathfrak{B}$ *such that for every B.t.* ϕ *defined in* $(\mathfrak{B}_0, I_0)$, $\bar{K}(\phi)$ *is defined in* $(\mathfrak{B}, I)$ *and* $F(\|\phi\|_{\mathfrak{B}_0, I_0}) = \|\bar{K}(\phi)\|_{\mathfrak{B}, I}$

Now, exactly as in the proof of 12.8, we have the following syntactical result (the extension of $\bar{K}$ to inequalities and sets of them is defined as for K).

15.5. Theorem. *For any set* $\Gamma \cup \{r\}$ *of inequalities* $\Gamma \vdash r$ *iff* $\bar{K}(\Gamma) \vdash \bar{K}(r)$.

A similar result for $\bar{\mathrm{QI}}$ (like 12.9(I)) follows directly from 15.1, so that Theorem 15.5 also "factors" through the predicate language.

§ 16. Embeddings in $\leqslant\aleph_\alpha$-generated B.a's

The natural generalization of 8.1 is this.

16.1. Theorem. *Let $\mathcal{B}_0$ be a B.a., $|\mathcal{B}_0| \leqslant \aleph_\alpha^{<\kappa}$. Then $\mathcal{B}_0$ has a complete embedding in a $(\leqslant\aleph_\alpha, <\kappa)$-generated B.a. $\mathcal{B}$ such that $|\mathcal{B}| = |\mathcal{B}_0|$.*

Proof. If $\mathcal{B}_0$ is finite or $\kappa = \aleph_0$ take $\mathcal{B} = \mathcal{B}_0$. If $\mathcal{B}_0$ is infinite and $|\mathcal{B}_0| \leqslant 2^{<\kappa}$ take $\mathcal{B}$ as in Theorem 8.1. So assume that $\kappa > \aleph_0$ and $|\mathcal{B}_0| = \lambda > 2^{<\kappa}$. Now if $\aleph_\alpha < \kappa$, then $|\mathcal{B}_0| \leqslant \aleph_\alpha^{<\kappa} = 2^{<\kappa}$. Therefore, the assumptions imply that $\aleph_0 < \kappa \leqslant \aleph_\alpha$ and $\kappa \leqslant \lambda \leqslant \aleph_\alpha^{<\kappa}$ where $\lambda = |\mathcal{B}_0|$. This can be treated in the same way as 8.1 using the $\aleph_\alpha^{<\kappa}$ independent sentences $\bar{p}_x$, $x \in \mathcal{P}_{<\kappa}(\aleph_\alpha)$. The algebra of formulas $\mathcal{B}$ is over

$$\{v_n \mid n < \omega\} \ ,$$

$$\{v_m \in v_n \mid m,n < \omega\} \cup \{P(v_n) \mid n < \omega\}$$
$$\cup \ \{v_m \approx \hat{\beta} \mid m < \omega, \beta < \aleph_\alpha\} \ .$$

We shall outline here an alternative proof, which is based solely on Definition 15.3 (with $\Omega = \omega$) and Theorem 15.5. Let A be a subset of $\mathcal{P}_{<\kappa}(\aleph_\alpha)$, $|A| = \lambda$. Such an A exists because $\lambda \leqslant \aleph_\alpha^{<\kappa} = |\mathcal{P}_{<\kappa}(\aleph_\alpha)|$. Let I_0 be a function from A onto $\mathcal{B}_0$, and let T_0 be the closure of $\{p_x \mid x \in A\}$ under $\neg$, $\wedge$, $\vee$. Clearly, each $\phi \in T_0$ is defined in $(\mathcal{B}_0, I_0)$. Let

$$\Gamma_1 = \{(\phi \leqslant \psi) \mid \phi, \psi \in T_0, \ \|\phi\| \leqslant \|\psi\|\}$$

($\|\cdot\|$ is short for $\|\cdot\|_{\mathcal{B}_0, I_0}$), and

$$\Gamma_2 = \left\{\phi \leqslant \vee X \mid \phi \in T_0, X \subseteq T_0, \|\phi\| = \bigvee_{\psi \in X}{}^{0} \|\psi\|\right\} .$$

Clearly $\Gamma_0 = \Gamma_1 \cup \Gamma_2$ is a set of inequalities true in $(\mathcal{B}_0, I_0)$, and hence for any $\phi, \psi \in T_0$, $\|\phi\| \leqslant \|\psi\|$ iff $\Gamma_0 \vdash \phi \leqslant \psi$. (It follows also that $\mathcal{B}_0$ is isomorphic to $T_0/\Gamma_0\vdash$.)

The role of Γ_2 will be to "capture" all joins in $\mathcal{B}_0$. Now let

$$Y_0 = \{\bar{K}(p_x) \mid x \in A\}$$

(this is a set of B.t's with atoms from

$$\{q_{mn} \mid m,n < \omega\} \cup \{r_{m\beta} \mid m < \omega, \beta < \aleph_\alpha\}$$
$$\cup \{s_n \mid n < \omega\}\ ,$$

because $A \subseteq \mathcal{P}(\aleph_\alpha)$). But $Y = \bigcup_{\phi \in Y_0} \mathrm{Sub}(\phi)$. T = closure of Y under $\neg, \wedge, \vee$. Thus T is closed under $\neg, \wedge, \vee$ and subterms, and $T \supseteq \{\bar{K}(\phi) \mid \phi \in T_0\}$. Let $\Gamma = \bar{K}(\Gamma_0)$. By the above and 15.5 we have, for all $\phi, \psi \in T_0$,

$$\|\phi\| \leqslant \|\psi\| \quad \text{iff} \quad \Gamma_0 \vdash \phi \leqslant \psi \quad \text{iff} \quad \Gamma \vdash \bar{K}(\phi) \leqslant \bar{K}(\psi),$$

hence the equation

$$F(\|\phi\|) = [\bar{K}(\phi)] \quad (\phi \in T_0)$$

where $[\cdot]$ is the equivalence class in $T/\Gamma\vdash$, defines an embedding F of $\mathcal{B}_0$ in $\mathcal{B} = T/\Gamma\vdash$. To see that F is complete let $b \in \mathcal{B}_0$, $C \subseteq \mathcal{B}_0$, $b = \bigvee^{\mathcal{B}_0} C$. Then for all $c \in C$, $(p_c \leqslant p_b) \in \Gamma_1$, hence $\Gamma_0 \vdash \bigvee_{c \in C} p_c \leqslant p_b$. On the other hand, $(p_b \leqslant \bigvee_{c \in C} p_c) \in \Gamma_2$. It follows that

$$\Gamma \vdash \bigvee_{c \in C} \bar{K}(p_c) \leqslant \bar{K}(p_b), \qquad \Gamma \vdash \bar{K}(p_b) \leqslant \bigvee_{c \in C} \bar{K}(p_c)\ .$$

It follows without difficulty that in $T/\Gamma\vdash$, $F(b) = \bar{K}(p_b)$ is the join of $\{F(c) \mid c \in C\}$. Thus the embedding F of $\mathcal{B}_0$ in $\mathcal{B}$ preserves joins and therefore meets.

To complete the proof we have to show that $\mathcal{B} = T/\Gamma\vdash$ has power λ and is $(\geqslant\aleph_\gamma, <\kappa)$-generated. Since in T occur at most $\aleph_\alpha$ distinct atomic B.t's and $\kappa \leqslant \lambda = |Y_0|$, an argument encountered already in §7–8 shows that it suffices to prove that each $\phi \in Y_0$ has length $<\kappa$, i.e., that if $x \in A$, $|\mathrm{Sub}(\bar{K}(p_x))| < \kappa$. Since $A \subseteq \mathcal{P}_{<\kappa}(\aleph_\alpha)$ and $\kappa > \aleph_0$, this follows directly from 15.3 (here $\Omega = \omega$), and the proof is complete. □

16.2. Remark. We have given the above proof not because it is simpler than a proof by the method of §8, but because it shows that the syntactical results 12.8 and 15.5 imply embedding theorems without going through the predicate-language. Later we shall see that these results, which deal only with B.t's, can be proved directly too.

The second main embedding theorem we have had is 14.1, and we may expect the following generalization (recall that ν is any cardinal while κ, for us, is always infinite):

16.3. Theorem. *If $\nu \leqslant \aleph_\alpha^{<\kappa}$ there is a $<\mathrm{cf}(\kappa)$-complete embedding of $\mathcal{F}^{<\mathrm{cf}(\kappa)}(\nu)$ in $\mathcal{F}^{<\kappa}(\aleph_\alpha)$. In particular, if κ is regular, $\nu \leqslant \aleph_\alpha^{<\kappa}$, then $\mathcal{F}^{<\kappa}(\nu)$ has a $<\kappa$-complete embedding in $\mathcal{F}^{<\kappa}(\aleph_\alpha)$.*

Proof. In proving this we again reduce quickly to the case $\aleph_0 < \kappa \leqslant \aleph_\alpha$ (note that the natural embedding of $\mathcal{F}^{<\kappa}(\aleph_0)$ in $\mathcal{F}^{<\kappa}(\aleph_\alpha)$ is $<\mathrm{cf}(\kappa)$-complete). Now, $\mathcal{F}^{<\kappa}(\aleph_\alpha)$ is isomorphic to $T/\emptyset\vdash$ where T is the set of all B.t's of length $<\kappa$ over the atoms $q_{mn}, r_{m\beta}, s_n$ $(m, n < \omega, \beta < \aleph_\alpha)$. This set T is closed under $\wedge$, $\vee$ applied to sets of $<\mathrm{cf}(\kappa)$ B.t's. We have already noted (in the proof of 16.1) that $\bar{K}(p_x)$ has length $<\kappa$ and so is a member of T when $x \in \mathcal{P}_{<\kappa}(\aleph_\alpha)$. So choose a set $A \subseteq \mathcal{P}_{<\kappa}(\aleph_\alpha)$ of power ν, and embed $\mathrm{BT}^{<\mathrm{cf}(\kappa)}(A)/\emptyset\vdash$ (which is isomorphic to $\mathcal{F}^{<\mathrm{cf}(\kappa)}(\nu)$) in $T/\emptyset\vdash$ by $[\phi] \to [\bar{K}(\phi)]$ $(\phi \in \mathrm{BT}^{<\mathrm{cf}(\kappa)}(A)$, and $[\cdot]$ is the equivalence class in $\mathrm{BT}^{<\mathrm{cf}(\kappa)}(A)/\emptyset\vdash$ or in $T/\emptyset\vdash)$. Clearly this is a $<\mathrm{cf}(\kappa)$-complete embedding, which proves the theorem. □

16.1 and 16.3 are the two basic theorems which contain all other quantitative results of this work as special cases or immediate corollaries. 16.1 is best possible in the same strong sense as 8.1. It implies the following generalization of 7.1.

16.4. Theorem. *If $\lambda \leqslant 2^{<\kappa}$, then there is a $(\leqslant\aleph_\alpha, <\kappa)$-generated B.a. of power λ containing λ disjoint elements. The possible infinite powers of $(\leqslant\aleph_\alpha, <\kappa)$-generated B.a's are exactly all λ such that $\lambda \leqslant \aleph_\alpha^{<\kappa}$.*

(For the proof use 16.1 with $\mathcal{B}_0$ the B.a. of finite and cofinite subsets of λ.)

From 16.3 we can deduce immediately that $|\mathcal{F}^{<\kappa}(\aleph_\alpha)| = \aleph_\alpha^{<\kappa}$ (but not that $\mathcal{F}^{<\kappa}(\aleph_\alpha)$ contains $\aleph_\alpha^{<\kappa}$ disjoint elements – this is not always true). Now let $\lambda \leqslant \aleph_\alpha^{<\kappa}$ and let $Y_0 = \{\phi_i \mid i < \lambda\}$ be a subset of $\mathrm{BT}^{<\kappa}(\aleph_\alpha)$ such that for $i \neq j$, ϕ_i and ϕ_j correspond to distinct elements of $\mathcal{F}^{<\kappa}(\aleph_\alpha)$ (i.e., $[\phi_i] \neq [\phi_j]$, in other words, it is not the case that $\vdash \phi_i \leqslant \phi_j$ and $\vdash \phi_j \leqslant \phi_i$). Let $Y = \bigcup_{\phi \in Y_0} \mathrm{Sub}(\phi)$ and T = closure of Y under $\neg$, $\wedge$, $\vee$. It is easy to see that if $\lambda \geqslant \kappa$ then $|T| = \lambda$ and $T/\emptyset\vdash$, which is isomorphic to a subalgebra of $\mathcal{F}^{<\kappa}(\aleph_\alpha)$, is $(\leqslant\aleph_\alpha, <\kappa)$-generated and contains the λ distinct elements $[\phi_i]$, $i < \lambda$. This proves in part the following generalization of 13.1.

16.5. Theorem. $|\mathcal{F}^{<\kappa}(\aleph_\alpha)| = \aleph_\alpha^{<\kappa}$, *and for each* $\lambda \leqslant \aleph_\alpha^{<\kappa}$, $\mathcal{F}^{<\kappa}(\aleph_\alpha)$ *has a* $(\leqslant\aleph_\alpha, <\kappa)$*-generated subalgebra of power* λ.

We have just proved this for the case $\kappa \leqslant \lambda$. But if $\lambda < \kappa$ then 13.2 applies and a subalgebra of $\mathcal{F}^{<\kappa}(\aleph_0)$ can also be viewed as a subalgebra of $\mathcal{F}^{<\kappa}(\aleph_\alpha)$.

As another corollary of 16.3 we have the generalization of Corollary 14.2.

16.6. Corollary. *For any cardinal* ν *and infinite cardinal* μ, $\mathcal{F}^{<\mu}(\nu)$ *has a* $<\mathrm{cf}(\mu)$*-complete embedding in* $\mathcal{F}^{<\kappa}(\aleph_\alpha)$ *whenever* $\mu \leqslant \mathrm{cf}(\kappa)$ *and* $\nu \leqslant \aleph_\alpha^{<\kappa}$.

The proof is just like that of 14.2. Thus all results of § 7, 8, 13, 14 generalize in the expected way, except the existence of many disjoint elements in free B.a's. It turns out more difficult to find interesting generalizations of the results of §§9–10 on complete B.a's, because the most obvious extensions (of 10.1, say) do not add anything to the original. On the whole, we can say much less about complete $\leqslant\aleph_\alpha$-generated B.a's for general α than for $\alpha = 0$. Many partial results can be given by extending the arguments of §§9–10 or by other methods (see [0, 27]) but no complete picture emerges. As an example of a partial result note that if $\mathfrak{B}$ is complete and $\leqslant\aleph_\alpha$-generated, $\kappa = \mathrm{CC}(\mathfrak{B})$, then clearly $2^{<\kappa} \leqslant |\mathfrak{B}| \leqslant \aleph_\alpha^{<\kappa}$, hence $|\mathfrak{B}| = 2^{<\kappa}$ if $\aleph_\alpha \leqslant 2^{<\kappa}$ (by 6.4(ii); κ is regular by Theorem 9.3).

§17. Disjoint elements in $\mathcal{F}^{<\kappa}(\aleph_\alpha)$

We know from §13 that $\mathcal{F}^{<\kappa}(\aleph_0)$, hence also $\mathcal{F}^{<\kappa}(\aleph_\alpha)$, has a subset of $2^{<\kappa}$ disjoint elements. The aim of this section is to prove that every set of disjoint elements from $\mathcal{F}^{<\kappa}(\aleph_\alpha)$ is of power $\leqslant 2^{<\kappa}$, so that using the CC notation of §9:

17.1. Theorem. $\mathrm{CC}(\mathcal{F}^{<\kappa}(\aleph_\alpha)) = (2^{<\kappa})^+$.

This will be proved by an adaptation to B.a's of a combinatorial technique known from the study of cardinals in Cohen extensions (cf.

[21, Lemma 10.3]. I am indebted to Menachem Magidor who showed me how this technique can be applied to answer the following problem: For any set A let ${}^A2 = \{f \mid f\colon A \to 2\}$ and let $F^{<\kappa}(A)$ where κ is any regular cardinal, be the $<\kappa$-complete field of subsets of ${}^A2$ generated by the sets $B_i = \{x \in {}^A2 \mid x(i) = 1\}$, $i \in A$. (When $A = \omega$ and $\kappa = \aleph_1$, $F^{<\kappa}(A)$ is the Borel field on ${}^\omega 2$). For singular cardinals put $F^{<\kappa}(A) = \bigcup_{\lambda<\kappa} F^{<\lambda^+}(A)$. The problem is to compute $\mathrm{CC}(F^{<\kappa}(\aleph_\alpha))$.

The main part of Magidor's solution consists of reducing everything to the following lemma and proving it. Following [21] we denote, for any sets A, B, $H_\kappa(A,B) = \{f \mid f$ is a function, $\mathrm{dom}(f) \in \mathcal{P}_{<\kappa}(A)$, $\mathrm{range}(f) \subseteq B\}$.

17.2. Lemma. *$H_\kappa(A, 2)$ has no subset of more than $2^{<\kappa}$ pairwise-incompatible elements.*

It suffices to prove the lemma for regular κ and the proof of [21, 10.3] works here with trivial modifications. The reader may look at that proof if he wants some motivation for what follows but in treating free B.a's instead of fields of sets some new ideas are needed. For completeness we state the result for fields of sets: If $\aleph_\alpha < \kappa$ then $\mathrm{CC}(F^{<\kappa}(\aleph_\alpha)) = (2^{\aleph_\alpha})^+$, and if $\aleph_\alpha \geqslant \kappa$ then $\mathrm{CC}(F^{<\kappa}(\aleph_\alpha)) = (2^{<\kappa})^+$.

The following proof of 17.1 is a simplification of my original proof using an idea of Saharon Shelah, Actually, when I told him I had proved 17.1 he immediately found an alternative proof, which is similar to the one given below but replaces the construction by transfinite recursion (or the sets A_β, Y_β) by the use of a general theorem of Erdös and Rado [2].

We have to prove that if $S \subseteq \mathcal{F}^{<\kappa}(\aleph_\alpha)$ is a set of pairwise-disjoint non-zero elements then $|S| \leqslant 2^{<\kappa}$.

Let S be such a set and let $Y \subseteq \mathrm{BT}^{<\kappa}(\aleph_\alpha)$ be a set containing one representative for each equivalence class in S. Thus $S = \{[\phi] \mid \phi \in Y\}$, and for all $\phi, \psi \in Y$, $\nvdash \phi \leqslant \mathrm{V}\emptyset$ but $\phi \neq \psi \Rightarrow\ \vdash \phi \wedge \psi \leqslant \mathrm{V}\emptyset$. We wish to prove that $|Y| \leqslant 2^{<\kappa}$.

First we need some notations concerning B.t's. Let $\mathrm{Supp}(\phi)$ (the support of ϕ) be $\{x \mid p_x$ occurs in $\phi\}$. In f is any function let $\phi(f)$ be the B.t. resulting from ϕ by substituting $p_{f(x)}$ for each occurrence of any p_x, $x \in \mathrm{dom}(f)$. If A is any set, let us say that ϕ and ψ have the same form over A and write $\phi \equiv_A \psi$ when there is a $1-1$ function f such that $\mathrm{dom}(f) = A \cup \mathrm{Supp}(\phi)$, f is the identity on A and $\psi = \phi(f)$ (hence $\mathrm{range}(f) = A \cup \mathrm{Supp}(\psi)$). Clearly this is an equivalence relation between B.t's.

17.3. Lemma. *If κ is regular, $|A| \leqslant 2^{<\kappa}$, and X is any set of B.t's each of length $<\kappa$, then $\equiv_A$ has at most $2^{<\kappa}$ equivalence classes on X.*

Proof. Let B be a set of power κ disjoint from A. Clearly for any B.t. ϕ of length $<\kappa$ there is a B.t. $\psi \in \mathrm{BT}^{<\kappa}(A \cup B)$ such that $\phi \equiv_A \psi$. Now $|\mathrm{BT}^{<\kappa}(A \cup B)| \leqslant |A \cup B|^{<\kappa}$ (this is proved similarly to the first part of 6.5; actually $|\mathrm{BT}^{<\kappa}(\nu)| = \nu^{<\kappa}$ always, but we do not need this). $|A \cup B| \leqslant 2^{<\kappa} + \kappa = 2^{<\kappa}$ and since κ is regular $(2^{<\kappa})^{<\kappa} = 2^{<\kappa}$. Thus $\mathrm{BT}^{<\kappa}(A \cup B)$, which contains representatives for all equivalence classes of $\equiv_A$ in X, has power $\leqslant 2^{<\kappa}$, which proves the Lemma. □

17.4. Lemma. *If $\phi \equiv_A \psi$ and* $\mathrm{Supp}(\phi) \cap \mathrm{Supp}(\psi) \subseteq A$ *and* $\vdash \phi \wedge \psi \leqslant \mathrm{V}\emptyset$, *then* $\vdash \psi \leqslant \mathrm{V}\emptyset$.

Proof. Let f be a $1-1$ function from $A \cup \mathrm{Supp}(\phi)$ onto $A \cup \mathrm{Supp}(\psi)$ such that $\psi = \phi(f)$ and $f \restriction A = \langle x \mid x \in A \rangle$. By the "rule of substitution" (the proof of which is obvious) $\vdash \phi \wedge \psi \leqslant \mathrm{V}\emptyset \Rightarrow \vdash \phi(f) \wedge \psi(f) \leqslant \mathrm{V}\emptyset$. Thus it suffices to show that $\psi(f) = \psi$. But $\psi(f) = \psi$ because if $x \in \mathrm{dom}(f)$ and p_x occurs in ψ then $x \in (A \cup \mathrm{Supp}(\phi)) \cap \mathrm{Supp}(\psi) \subseteq A$ and so $f(x) = x$. □

Having proved the two lemmas, we return to the set $Y \subseteq \mathrm{BT}^{<\kappa}(\aleph_\alpha)$ mentioned above. We assume that κ is regular.

Define by recursion on β a sequence (A_β) of subsets of $\aleph_\alpha$ and a sequence (Y_β) of subsets of Y as follows:

$A_0 = Y_0 = \emptyset$ and for limit ordinals δ, $A_\delta = \bigcup_{\beta<\delta} A_\beta$, $Y_\delta = \bigcup_{\beta<\delta} Y_\beta$. Suppose A_β and Y_β have been defined. From each equivalence class of $\equiv_{A_\beta}$ on Y choose one representative. Let $Y_{\beta+1} = \bigcup Y_\beta$ be the set of these representatives, and $A_{\beta+1} = \bigcup_{\phi \in Y_{\beta+1}} \mathrm{Supp}(\phi)$. It is clear that (A_β), (Y_β) are both increasing sequences (w.r.t. inclusion) and for all β, $A_\beta = \bigcup_{\phi \in Y_\beta} \mathrm{Supp}(\phi)$. The basic property of these sequences is this:

$$(\forall\beta)\,(\forall\phi \in Y)\,(\exists\psi \in Y_{\beta+1})\quad (\phi \equiv_{A_\beta} \psi).$$

We now prove by induction that for all $\beta \leqslant \kappa$ (or even $\beta \leqslant 2^{<\kappa}$), $\max(|A_\beta|, |Y_\beta|) \leqslant 2^{<\kappa}$. The case of 0 or limit ordinals is trivial, and so it remains to prove only that if $\max(|A_\beta|, |Y_\beta|) \leqslant 2^{<\kappa}$, then $\max(|A_{\beta+1}|, |Y_{\beta+1}|) \leqslant 2^{<\kappa}$. Assume that $|A_\beta| \leqslant 2^{<\kappa}$, $|Y_\beta| \leqslant 2^{<\kappa}$. By Lemma 17.3, $\equiv_{A_\beta}$ has at most $2^{<\kappa}$ equivalence classes on Y, so $|Y_{\beta+1}| \leqslant |Y_\beta| + 2^{<\kappa} = 2^{<\kappa}$. Now if $\phi \in Y_{\beta+1}$, then (ϕ has length $<\kappa$) which implies that $|\mathrm{Supp}(\phi)| < \kappa$; it follows that

$$|A_{\beta+1}| = |\bigcup_{\phi \in Y_{\beta+1}} \mathrm{Supp}(\phi)| \leqslant \kappa \cdot |Y_{\beta+1}| \leqslant \kappa \cdot 2^{<\kappa} = 2^{<\kappa}.$$

This completes the induction, and so $|Y_\beta| \leqslant 2^{<\kappa}$ for all $\beta \leqslant \kappa$. Note that $A_\kappa = \bigcup_{\beta<\kappa} A_\beta$. Let $\phi \in Y$. By the regularity of κ and the fact that $|\mathrm{Supp}(\phi)| < \kappa$ there is some $\beta < \kappa$ such that $\mathrm{Supp}(\phi) \cap A_\kappa \subseteq A_\beta$. Take $\psi \in Y_{\beta+1}$ so that $\phi \equiv_{A_\beta} \psi$. Then $\mathrm{Supp}(\phi) \cap \mathrm{Supp}(\psi) \subseteq \mathrm{Supp}(\psi) \cap A_\kappa \subseteq A_\beta$, so by Lemma 17.4 (for A_β) $\vdash \phi \wedge \psi \leqslant \bigvee\emptyset$ implies $\vdash \psi \leqslant \bigvee\emptyset$. But $\phi, \psi \in Y$ so by the choice of Y this implies $\phi = \psi$, hence $\phi \in Y_{\beta+1} \subseteq Y_\kappa$. The conclusion is that $Y = Y_\kappa$ and so $|Y| \leqslant 2^{<\kappa}$, for regular κ.

If κ is singular then for any $\lambda < \kappa$ the above shows that Y contains at most 2^λ B.t's of length $<\lambda^+$, and hence $|Y| \leqslant \Sigma_{\lambda<\kappa} 2^\lambda = 2^{<\kappa}$. This completes the proof. □

§18. Bypassing the predicate language

The aim of this section is to outline an alternative approach to the subject of this paper, which is based on syntactical considerations concerning B.t's and avoids the introduction of the predicate language. (See note (5) on p. 427).

One begins by defining B.t's, valuations, free B.a's, derivations of inequalities between B.t's and the B.a's $T/\Gamma \vdash$, and proving their basic properties as in §11. Then the B.t's π_x^N and ρ_x^0 are introduced by 12.3. Their definition can be motivated by the following two lemmas (cf. §3, Lemma 3.3 and Theorem 3.4), the first of which justifies calling the π_x^N's locating (propositional) formulas.

18.1. Lemma. *Let f be a 1–1 function from Ω onto a transitive set t. Consider any valuation $(\mathcal{B}, I)$ in which*

$$\|q_{MN}\| = \begin{cases} 1^{\mathcal{B}} & \text{if } f(M) \in f(N), \\ 0^{\mathcal{B}} & \text{otherwise} \end{cases}$$

(for all $M, N \in \Omega$). Then for each $N \in \Omega$ and each x,

$$\|\pi_x^N\| = \begin{cases} 1^{\mathcal{B}} & \text{if } f(N) = x, \\ 0^{\mathcal{B}} & \text{otherwise.} \end{cases}$$

18.2. Lemma. *Let f and t be as above, and let $(\mathcal{B}_0, I_0)$ be a valuation such that* $\mathrm{dom}(I_0) \subseteq t$. *Consider a valuation $(\mathcal{B}_0, I)$ such that, for all $M, N \in \Omega$,*

$$\|q_{MN}\|_I = \begin{cases} 1 & \text{if } f(M) \in f(N), \\ 0 & \text{if } f(M) \notin f(N), \end{cases}$$

and whenever $f(M) \in$ $\mathrm{dom}(I_0)$, *$\|r_M\|_I = I_0(f(M))$ (such an I, though not unique, clearly exists). Then for any $x \in$* $\mathrm{dom}(I_0)$, *$I_0(x) = \|\rho_x^0\|_I$. Hence, for any B.t. ϕ defined in $(\mathcal{B}_0, I_0)$, $\|\phi\|_{I_0} = \|K(\phi)\|_I$ (where K is the translation defined in* §12).

The proof of Lemmas 18.1 and 18.2 is rather straightforward and left to the reader (there is no need to go through the predicate language). 18.2 is somewhat awkward in formulation, but it really asserts that if one looks at K as a translation from (an arbitrary) propositional language to one based on the atoms q_{MN}, r_N $(M, N \in \Omega)$, then any Boolean model for the former can be interpreted by K in a model for the latter with values in the same B.a., provided only that a $1-1$ correspondence of Ω and some large enough transitive set can be found.

Now suppose that Ω is infinite countable, and let us restrict all sets temporarily to HC $= H(\aleph_1)$ ($= \{x \mid \mathrm{TC}(x)$ is countable$\}$), so that any two infinite sets have a $1-1$ correspondence. Then 18.1 implies that if $(M, x) \neq (N, y)$ $(M, N \in \Omega)$, then the inequality $\pi_x^M \leqslant \pi_y^N$ is false in some two-valued model (i.e., valuation in the B.a. $\bar{2} = \{0 \leqslant 1\}$), hence $\not\vdash \pi_x^M \leqslant \pi_y^N$. A similar argument, based on 18.2 proves that 12.8 is true when $(\Gamma \cup \{r\}) \in$ HC.

The problem now is how to prove 12.7 and 12.8 in full generality, i.e., without countability assumptions. This can be done by means of a set-theoretic meta-theorem due to Lévy ([13, Theorem 36]). The special case we need is this:

Let ϕ be a Σ_1-formula (of set theory). Then

$$(\exists x_1, \ldots, x_n)(\phi)(x_1, \ldots, x_n) \Rightarrow (\exists x_1, \ldots, x_n \in \mathrm{HC})(\phi)(x_1, \ldots, x_n)$$

is a theorem of ZF (Lévy proved this for ZF + dependent choices, but it is known to hold for ZF too; actually there is no special reason to avoid AC here.)

To apply this theorem we must assume that B.t's, inequalities and derivations have been defined in a natural way within set theory. For instance take

$$p_x = (0, x), \quad \neg\phi = (1, \phi), \quad (\phi \leqslant \psi) = (4, \phi, \psi)$$
$$\wedge X = (2, X), \quad \vee X = (3, X),$$

and define derivations as sequences or trees in some definite way. Also in 12.7, 12.8 there is no loss of generality in assuming for definiteness that $q_{MN} = p_{(0, M, N)}$, $r_N = p_{(1, N)}$ ($M, N \in \Omega$). We assert that the statements "Ω, N, x, y afford a counter-example to 12.7", "Ω, Γ, r afford a counter-example to 12.8" are both equivalent in ZF to Σ_1-formulas. This is rather easily seen from the known closure properties of Σ_1^{ZF}-formulas (see [13] or [11]), once one proves that "$\Gamma \vdash r$" and "$\Gamma \nvdash r$" are both Σ_1^{ZF}. Assuming this, the proof of 12.7 and 12.8 is complete, because we already know that there is no counterexample in HC, and can apply Lévy's theorem.

Now, the Σ_1^{ZF}-ness of "$\Gamma \vdash r$" is obvious because "$\Gamma \vdash r$" means the existence of a derivation of r from Γ. The Σ_1^{ZF}-ness of "$\Gamma \nvdash r$" is also easy to see, because by the completeness theorem it is equivalent to the existence of a valuation in which Γ is true and r false. [The same remarks are valid for predicate logic, because models with values in incomplete B.a's can be allowed; while writing the final version of this paper I found that this fact had also been observed by Gregory [6, Lemma 3.1, p.453]; an alternative proof of the Δ_1^{ZF}-ness of "$\vdash$" follows from Barwise's completeness theorem.]

Thus we see that 12.7 and 12.8 are proved without using the predicate language. 15.5 can be proved in the same way – first directly for the case $\Omega, \Gamma, r \in$ HC and then by Lévy's theorem in the general case. The considerations of § 16 now make clear that all the theorems of previous sections that do not deal explicitly with predicate-formulas are provable on the basis of 12.7, 12.8 and 15.5 in so far as the original proofs are not already in terms of B.t's (as in § 13, 14, 17).

Thus for all algebraic results Lévy's theorem can take the place of the predicate language. Note that the relative importance of syntax (versus semantics) is greater in this approach. Whereas previously one could view "$\Gamma \vdash r$" simply as short for "r is a Boolean consequence of Γ", here one needs a syntactical definition of it to verify the Σ_1-ness.

CHAPTER II

EXTENSION TO PARTIALLY ORDERED STRUCTURES; CONNECTIONS WITH FORCING AND APPLICATIONS

We are going to present the basic idea of §4 in a new form. Instead of talking about equivalence classes of formulas w.r.t. the relation $\equiv^M$ ($\phi \equiv^M \psi$ iff $\|\phi[f]\|_M = \|\psi[f]\|_M$ for all assignments f into the Boolean-valued model M), we represent each formula ϕ (with free variables from $\{v_n \mid n < \omega\}$) by the function $\tilde{\phi}$ which takes to $\tilde{\phi}(f) = \|\phi[f]\|_M$ each assignment f: $\{v_n \mid n < \omega\} \to M$. Thus $\phi \equiv^M \psi$ iff $\tilde{\phi} = \tilde{\psi}$. In this way it is seen that the countably generated B.a. $\mathcal{B}$ in which $\mathcal{B}_0$ is completely embedded can be viewed as an algebra of functions into $\mathcal{B}_1$ (the normal completion of $\mathcal{B}_0$), and a closer examination shows that functions into $\mathcal{B}_0$ suffice. Thus, $\mathcal{B}$ can be taken as a subalgebra of a direct power $\mathcal{B}_0^W$ of $\mathcal{B}_0$, consisting of functions ξ defined on the space W of all assignments, such that $\xi(x)$ depends on only finitely many coordinates of the assignment x (i.e., on the values assigned by x to finitely many variables).

We generalize this basic idea in two ways: Firstly, B.a.'s are replaced by arbitrary posets (partially ordered sets), and secondly $\{v_n \mid n < \omega\}$ is replaced by $\{v_N \mid N < \Omega\}$ where Ω is any regular cardinal, and one considers functions which depend on $<\Omega$ coordinates of the assignment. The representation by means of $\mathcal{L}_{\infty\omega}$-formulas shows that the resulting poset of functions is $\leqslant\Omega$-generated. We also show how to handle additional relations and operations with which the poset may be equipped, so the final result (22.1) is a very general embedding theorem with cardinality bounds (the reader can find the exact formulation there – it is too long to reproduce here). As special cases or easy corollaries of 22.1 we can get our previous embedding result for B.a's (16.1), and the same result for lattices, modular lattices, Stone algebras, etc. We also conclude that if Ω is regular, $\nu_1, \nu_2 \leqslant \Omega$ then every $<\Omega$-complete $(<\nu_1, <\nu_2)$-distributive lattice (or B.a.) has a complete embedding in a $\leqslant\Omega$-generated lattice (B.a.) of the same kind (§23).

In §24 we try to extend those results of Chapter I which refer not to a single B.a. but to the class of all B.a's (e.g., results on free algebras and the translation result 12.8). We introduce the "lattice-τ-terms" which are analogous to B.t's, use them to define free $<\kappa$-complete algebras of type τ and show that under certain assumptions on the class under consideration most of the results obtained for the class of B.a's remain valid To be sure, the situation is not yet completely clear, and some basic problems are not even touched upon.

§§25–26 explain the connection of our approach with the one based on forcing (for the case of B.a's) and use forcing to prove the following generalization of Kripke's theorem: Let Ω be regular. Each $(<\Omega, <\infty)$-distributive B.a. has a complete embedding in some complete, $\leqslant\Omega$-generated $(<\Omega, <\infty)$-distributive B.a.

(The corresponding generalization of the Gaifman–Hales theorem is known – it was found by Gaifman and Hales themselves.)

We could also get cardinality estimates here, but this task is now routine and left to the reader (perhaps finding *best possible* estimates is non-trivial).

In §27 we present some possible directions of further development, and an example of the application of our methods to infinitary logic on models of weak set theories.

In general the exposition continues to be self-contained as in Chapter I, except that we have had to assume some knowledge of, and preferably also previous experience with, forcing in §26 and primitive-recursive set functions in §27. Explaining these matters would increase the length of the chapter too much, but detailed references are given.

Since the subject of this chapter contains some unexplored domains, there was no point in giving a very detailed exposition of the kind given in Chapter I, or of checking if each result is best possible as done there. Rather we have contented ourselves with a less formal presentation, and have left proofs whose ideas are clear from Chapter I to the reader. This also helped to save time and space, and a more complete presentation had better wait until the study reaches a more advanced stage.

The notations and terminology are those of Chapter I, with some obvious extensions of various concepts from B.a's to partially ordered structures. Sometimes we denote a structure (set with relations and operations) by a script letter (e.g. $\mathcal{Q}$) and the underlying set by the corresponding italic letter (Q), but we shall not be very consistent in this usage (especially in the case of B.a's or of posets without additional structures).

§ 19. Poset-valued models

Given a language $\mathcal{L}$, the $\mathcal{L}_{\infty\omega}$-formulas constructed from atomic ones by $\wedge$, $\vee$, $\forall$, $\exists$ only are called $\mathcal{L}_{\infty\omega}{}^{+}$-formulas (by adding names for the elements of a set A, we get the $\mathcal{L}_{\infty\omega}{}^{+}(A)$-formulas). Let Q be a poset, $\mathcal{L}$ a language. A Q-valued model M for $\mathcal{L}$ is defined like an ordinary model except that an n-place predicate $R \in \mathcal{L}$ is interpreted by a function $R^M : M^n \to Q$. In such a model M, $\mathcal{L}_{\infty\omega}{}^{+}(M)$-sentences can be evaluated as in Boolean-valued models, interpreting $\wedge$, $\forall$ by $\wedge^Q$ (meet in Q) and $\vee$, $\exists$ by $\vee^Q$ (join in Q). However, some meets and joins may fail to exist in Q, and we therefore assign an improper value $*$ ($\notin Q$) in the "undefined" case, as is done for B.t's in § 11. An $\mathcal{L}_{\infty\omega}{}^{+}(M)$-formula ϕ is said to be strongly defined in M when for every assignment f of values in M to its free variables, $\phi[f]$ is defined in M (i.e., $\|\phi[f]\|_M \in Q$).

Note that:

(1) every atomic $\mathcal{L}(M)$-formula is strongly defined in M;

(2) if $\wedge X$ or $\vee X$ is strongly defined, so is every member of X;

(3) if $(\forall u)(\psi)$ or $(\exists u)(\psi)$ is strongly defined, so is $\psi(\genfrac{}{}{0pt}{}{u}{v})$ for every unbindable variable v.

When $Q = \{0 \leqslant 1\}$ is the two-element poset, Q-valued models are called two-valued and can be identified with ordinary models, so that $M \models \phi$ iff $\|\phi\|_M = 1$.

So much for generalities. Now consider a fixed poset Q which is given as an indexed set $Q = \{q_i \mid i \in A\}$ where A is transitive, and assume that Q has extreme elements $0, 1$ ($0 \leqslant x \leqslant 1$ for all $x \in Q$). This implies $A \neq \emptyset$ and we consider the following Q-valued model M for $\mathcal{L} = \{\epsilon, \bar{\epsilon}, \boldsymbol{P}\}$ (ϵ, $\bar{\epsilon}$ are binary predicates, $\boldsymbol{P}$ a unary predicate). $M = \langle A, \epsilon^M, \bar{\epsilon}^M, \boldsymbol{P}^M\rangle$ where for $x, y \in A$:

$$\epsilon^M(x, y) = \begin{cases} 1 & x \in y, \\ 0 & \text{otherwise,} \end{cases}$$

$$\bar{\epsilon}^M(x,y) = \begin{cases} 0 & x \in y, \\ 1 & \text{otherwise,} \end{cases} \qquad P(x) = q_x \,.$$

We define the locating formulas π_x (cf. 2.1) as follows:

$$\pi_x(v_0) = (\forall u)\,(u \mathbin{\bar{\epsilon}} v_0 \vee \bigvee_{y\in x} \pi_y(u)) \wedge \bigwedge_{y\in x} (\exists u)\,(u \mathbin{\epsilon} v_0 \wedge \pi_y(u)).$$

As in §3 we put $\rho_x = (\exists u)\,(\pi_x(u) \wedge P(u))$. It is easily verified that in the model M, $\|\pi_x(\hat{a})\| = 1$ if $x = a$, $= 0$ otherwise (any x and any $a \in A$), $\|\rho_x\| = q_x$ if $x \in A$ (and $\|\rho_x\| = 0$ otherwise).

§20. The poset C of cylindric functions

Let $Q = \{q_i \mid i \in A\}$ (A transitive) be a poset with 0, 1 and let Ω be a regular cardinal. The Q-valued model M for $\mathcal{L} = \{\epsilon, \bar{\epsilon}, P\}$ has been defined in §19. An $\mathcal{L}_{\infty\omega}{}^{+}$-formula ϕ is called, in this context, admissible when the set of free variables of ϕ is a subset of $\{v_N \mid N < \Omega\}$ whose cardinality is $<\Omega$, and ϕ is strongly defined in M. We use the notation Sub*(ϕ) (as defined in 5.2) in the Z-sense where $Z = \{v_N \mid N < \Omega\}$ (we use $N, N', N_0, \ldots$ as variables over Ω). If ϕ is admissible and $\psi \in$ Sub*(ϕ) then ψ is admissible.

Put $W = \{x \mid x : \Omega \to A\}$. Regard $Q^W = \{\xi \mid \xi : W \to Q\}$ as a poset (a direct power of Q), and for each $q \in Q$ let q^* be the constant function $W \to \{q\}$ Then $*$ is a complete embedding of Q in Q^W and we put $Q^* = \{q^* \mid q \in Q\}$. $\xi \in Q^W$ is called cylindric when $(\exists N_0 < \Omega)\,(\forall w, x \in W)$ $[w \mid N_0 = x \mid N_0 \Rightarrow \xi(w) = \xi(x)]$. Put $C = \{\xi \in Q^W \mid \xi$ is cylindric$\}$. Clearly $Q^* \subseteq C$.

For each admissible formula ϕ let $\tilde{\phi} : W \to Q$ be defined by $\tilde{\phi}(x) = \|\phi[x]\|_M$ ($x \in W$), where x is identified here with the assignment $v_N \mapsto x(N)$ ($N < \Omega$) into M. Since ϕ is admissible, $\tilde{\phi}$ is well-defined and cylindric. Also, $\tilde{\rho}_i$ is the constant function q_i^* ($i \in A$), where ρ_i is defined at the end of §19.

Consider now any set T of admissible formulas such that $\{\rho_i \mid i \in A\} \subseteq T$ and $(\forall \phi \in T)$ (Sub*(ϕ) $\subseteq T$). Put $\tilde{T} = \{\tilde{\phi} \mid \phi \in T\}$ and then $Q^* \subseteq \tilde{T} \subseteq C$, and $*$ is a complete embedding of Q in $\tilde{T}$ (the latter being regarded as a sub-poset of Q^W). We observe that for each admissible formula ϕ:

(I) If $\phi = \bigwedge X$ then $\tilde{\phi} = \bigwedge^C_{\psi\in X} \tilde{\psi}$, and in fact $\tilde{\phi} = \bigwedge^{Q^W}_{\psi\in X} \tilde{\psi}$; dually for $\bigvee X$.

(II) If $\phi = (\forall v)(\psi)$ then $\widetilde{\phi} = \bigwedge^{C}_{N<\Omega} \psi(^{u}_{v_N})$ and dually for $(\exists u)(\psi)$ (here the meet in C is not necessarily the meet in Q^W).

The proofs are easy but in (II) one uses the restriction on the number of free variables of ϕ.

It follows by induction on the depth of formulas $\phi \in T$ that $\{\widetilde{\phi} \mid \phi \in T, \phi$ is atomic$\}$ generates the poset $\widetilde{T}$ by means of meets and joins existing in $\widetilde{T}$ which are actually meets (joins resp.) also in the larger poset C. Thus $\widetilde{T}$ is $\leqslant\Omega$-generated and * is a complete embedding of Q in T.

We now turn to some interesting choices of T:

(0) $$T_0 = \bigcup_{i\in A} \mathrm{Sub}^*(\rho_i)\,.$$

This is the smallest T satisfying our assumptions. Note that ρ_i, $\pi_i(v_N)$, $P(v_N) \in T_0$ for all $i \in A$, $N < \Omega$. Let

(1) $$P = \{p \mid (\exists N_0 < \Omega)\ [p : N_0 \to A]\}$$

For each $p \in P$ and $i \in A$ consider the formula $\phi_{p,i} = \bigwedge(\{\pi_{p(N)}(v_N) \mid N \in \mathrm{dom}(p)\} \cup \{\rho_i\})$. Under any assignment $x \in W$ we have:

$$\begin{aligned}\|\phi_{p,i}[x]\|_M &= \bigwedge^Q [\{\|\pi_{p(N)}(x(N))\|_M \mid N \in \mathrm{dom}(p)\} \cup \{\|\rho_i\|_M\}]\\ &= \begin{cases} q_i & \text{if } p \subseteq x\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus if we take $T_1 = T_0 \cup \{\phi_{p,i} \mid p \in P, i \in A\}$, T_1 satisfies the assumptions on T and $\widetilde{T}_1$ contains, for each $p \in P$, $i \in A$ the function

$$\widetilde{\phi}_{p,i} : x \mapsto \begin{cases} q_i & p \subseteq x\\ 0 & \text{otherwise}\end{cases} \qquad (x \in W).$$

The interest in T_1 stems from the following observation: The set $B = \{\widetilde{\phi}_{p,i} \mid p \in P, i \in A\}$ is a basis for C in the sense that each $\xi \in C$ is a join ($\bigvee^C$) of elements of B. Indeed let $\xi \in C$ and choose $N_0 < \Omega$ such that $\xi(x)$ depends only on $x \restriction N_0$ (for $x \in W$). For each $p : N_0 \to A$ let $\xi(p)$ be the common value $\xi(x)$ for all $x \supseteq p$. Clearly $\xi(x) = \bigvee^Q \{\widetilde{\phi}_{p,i}(x) \mid p : N_0 \to A, q_i = \xi(p)\}$ for each $x \in W$. Therefore putting $B' = \{\widetilde{\phi}_{p,i} \mid p : N_0 \to A, q_i = \xi(p)\}$ we have $B' \subseteq B$ and $\xi = \bigvee^{Q^W} B' = \bigvee^C B'$. Thus B, and hence T_1, is a basis for C. The argument also shows that putting $\phi = \bigvee\{\phi_{p,i} \mid p : N_0 \to A, q_i = \xi(p)\}$ we have $\xi = \widetilde{\phi}$. Note that ϕ is admissible and has the form $\bigvee X$ for some $X \subseteq T_1$. This leads to our next choice of T.

(2) $\quad T_2 = T_1 \cup \{\bigvee X \mid X \subseteq T_1 \text{ and } \bigvee X \text{ is admissible}\}.$

T_2 satisfies our assumptions on T and by the last observation $\widetilde{T}_2 = C$.

To sum up: $Q^* \subseteq \widetilde{T}_0 \subseteq \widetilde{T}_1 \subseteq \widetilde{T}_2 = C$, each $\widetilde{T}_i$ $(i = 0, 1, 2)$ is generated by $\{\widetilde{\phi} \mid \phi \in T_0$ and ϕ is atomic$\}$ (whose power is $\leqslant \Omega$) by means of the partial operations $\bigwedge^C, \bigvee^C$ (without going out of $\widetilde{T}_i$), and hence by means of $\bigwedge^{\widetilde{T}_i}, \bigvee^{\widetilde{T}_i}$. Also $\widetilde{T}_1$ is a basis for C (that is, generates C by means of $\bigvee^C$ alone).

Let us note another property of C: If $D \subseteq C$, $|D| < \Omega$ and D has a meet ξ_0 in Q^W then $\xi_0 \in C$ (and thus $\xi_0 = \bigwedge^C D$). Indeed we must have $\xi_0(x) = \bigwedge^Q \{\xi(x) \mid \xi \in D\}$ for each $x \in W$, and clearly if each $\xi \in D$ is cylindric and $|D| < \Omega$ then ξ_0 is cylindric (regularity of Ω is used here). An analogous remark applies to joins. If, in particular, Q is $<\Omega$-complete (i.e., every subset of Q of power $<\Omega$ has a meet and a join in Q), then so is Q^W hence so is C. Also, if Q is $<\aleph_0$-complete (that is, a lattice) then C is $<\aleph_0$-complete too.

§21. Extending relations and operations from Q to C

We now consider the situation in which our poset Q is equipped with additional, possibly infinitary, relations and operations, thus becoming a partially ordered structure $\mathcal{Q}$. First let us clarify our terminology.

By a type of structure we mean any set τ of relation and operation symbols, but now these symbols are taken to be R_i^η, O_i^η for any set (usually ordinal) η and any i (in §5 η was required to be a natural number). η serves as the index set for the "empty places" (arguments) of the relation or operation. In a structure $\mathcal{Q}$ of type τ each $r = R_i^\eta \in \tau$ is interpreted as a relation $r^{\mathcal{Q}} \subseteq Q^\eta$ and each $o = O_i^\eta \in \tau$ as an operation $o^{\mathcal{Q}} : Q^\eta \to Q$ where Q is the domain of individuals (which we usually denote by the italic capital letter corresponding to the script letter used for the structure).

If τ contains the two particular predicates $\approx$ and $\leqslant$ (R_0^2 and R_1^2, say) we call τ a type of partially ordered structures. A structure $\mathcal{Q}$ of such a type is said to be a partially ordered structure (p.o.s.) when $\approx^{\mathcal{Q}}$ is the equality relation on Q, and $\leqslant^{\mathcal{Q}}$ is a partial order. When $\mathcal{Q}$ is a p.o.s. we shall denote $\leqslant^{\mathcal{Q}}$ also by $\leqslant^Q$, and meet and join in $\leqslant^Q$ by $\bigwedge^Q$, $\bigvee^Q$. When $\bigwedge^Q A$, $\bigvee^Q A$ exist for every $A \subseteq Q$ [of power $<\nu$] we say that $\mathcal{Q}$ is a complete [$<\nu$-complete] p.o.s. Q is 0-complete (i.e., <1-complete) iff $\leqslant^Q$ has extreme elements 0^Q, 1^Q, and we then say also the p.o.s. Q is bounded (in all these notations the superscript ^{Q} is often omitted).

The notions of isomorphism, substructure, direct product, homomorphism and embedding are defined for structures (of any type τ) as usual (see, e.g., [4, §36]). Note that an embedding is required to preserve relations in both directions, so that a $1-1$ homomorphism is not always an embedding. In the case of p.o.s's we also have the notions of a $<\nu$-substructure (one closed under meets and joins of $<\nu$ elements existing in the original p.o.s.), a $<\nu$-complete homomorphism or embedding (called complete when $\nu = \infty$) and variants of these.

We mention that if Q is any structure and W a set, then * defined in §20 is an embedding of Q in the direct power Q^W, and in case Q is a p.o.s. so is Q^W and $*$ is a complete embedding. It follows that in this case if $\mathcal{C}$ is any substructure of Q^W and $Q^* \subseteq C$, then $*$ is a complete embedding of Q in $\mathcal{C}$.

From now on we consider the following situation:

(1) Q is a bounded p.o.s.,

(2) $Q = \{q_i \mid i \in A\}$, A transitive,

(3) Ω is a regular cardinal, and each operation of Q is a $<\Omega$-ary (i.e., if $O_i^\eta \in$ (type of Q), then $|\eta| < \Omega$).

In §20 we have defined the set W and the set $C \subseteq Q^W$ of cylindric functions (actually $\langle C; \leqslant^C\rangle$ is a $<\Omega$-subposet of $\langle Q; \leqslant^Q\rangle^W$). We note that C is closed under the operations of Q^W, and hence determines a substructure $\mathcal{C}$ of Q^W. This follows easily from the fact that if $|\eta| < \Omega$ and ξ_i is cylindric for each $i \in \eta$, then by the regularity of Ω there is an $N < \Omega$ such that, for $x, y \in W$, $x \restriction N = y \restriction N$ implies $(\forall i \in \eta)\,(\xi_i(x) = \xi_i(y))$. Thus given the bounded p.o.s. Q we have not only a poset but a bounded p.o.s. $\mathcal{C}$ of the same type in which Q is completely embedded and which is $\leqslant\Omega$-generated already as a poset. Note that if $\nu < \Omega$ and $\langle Q; \leqslant^Q\rangle$ is $\leqslant\nu$-complete, one is free to enrich Q by the two ν-ary operations of meet and join. Then the corresponding operations induced on C will be the ν-ary meet and join of $\langle C; \leqslant^C\rangle$ (and in particular $\mathcal{C}$ is $\leqslant\nu$-complete, but this we already know).

In §20 we have defined the subsets $\widetilde{T}_0 \subseteq \widetilde{T}_1 \subseteq \widetilde{T}_2 = C$. $\widetilde{T}_0$ and $\widetilde{T}_1$ will not, in general, be closed under the operations of $\mathcal{C}$. However, let $C_0 \subseteq C$ be the closure of $\widetilde{T}_0$ under:

(1) meets and joins of $<\Omega$ element which exist in $\mathcal{C}$ (or even only those which exist in Q^W and belong to C).

(2) the operations of $\mathcal{C}$.

It is easily seen from §20 that $C_0 \supseteq \widetilde{T}_1$ and C_0 is generated by $\{\widetilde{\phi} \mid \phi \in T_0, \phi \text{ atomic}\}$ by means of meets and joins existing in $\mathcal{C}$. Also, C_0 is the domain of a substructure $\mathcal{C}_0$ of $\mathcal{C}$, and $\mathcal{C}_0$ (like $\mathcal{C}$, is a $<\Omega$-substructure of Q^W. Thus $\mathcal{C}_0$ has all the main properties of $\mathcal{C}$ (and is more "economic" than $\mathcal{C}$ when one comes to cardinality estimates).

§22. Generalization to $\aleph_\alpha$ generators and cardinality estimates

We have seen how to embed the structure Q in a $\leqslant\Omega$-generated structure $\mathcal{C}$. Now suppose $\aleph_\alpha \geqslant \Omega$ and one wants to embed Q in a $\leqslant\aleph_\alpha$-generated structure (in the hope of reducing the complexity of generation as compared to the case of $\leqslant\Omega$ generators). The procedure of §§15–16 can be followed, or one can give a unified treatment of all cases by using locating formulas in a set-theoretic universe with individuals ("urelementen"). It is not expedient to "simplify" things by taking $\aleph_\alpha$ as the new value of Ω, because this would lead to an unnecessarily large $\mathcal{C}$ or $\mathcal{C}_0$. Since no new ideas beyond those of §15 are involved, we shall present the main result (Theorem 22.1) in the most general form without proof.

Before giving the result we shall explain our terminology concerning the complexity of generation. Let $\mathcal{C}$ be a p.o.s., and let $G \subseteq C$. By $[G]_{\mathcal{C}}^{<\kappa}$ we mean, for regular κ, the closure of G under meets and joins of $<\kappa$ elements existing in $\mathcal{C}$ and those operations of $\mathcal{C}$ having $<\kappa$ places. For singular κ we put $[G]_{\mathcal{C}}^{<\kappa} = \bigcup_{\lambda<\kappa} [G]_{\mathcal{C}}^{<\lambda^+}$, and this holds for regular $\kappa > \aleph_0$ as well. When $[G]_{\mathcal{C}}^{<\kappa} = C$ we say that G *generates* $\mathcal{C}$ *in the* $<\kappa$*-sense*. Let us also say that G generates $\mathcal{C}$ in the *weak* $<\kappa$-sense when C is the closure of G under meets and joins of $<\kappa$ elements existing in $\mathcal{C}$ and all the operations of $\mathcal{C}$ (thus for singular κ, G generates $\mathcal{C}$ in the weak $<\kappa$-sense iff G generates $\mathcal{C}$ in the weak $<\kappa^+$-sense). The expressions "$\mathcal{C}$ is $(<\nu, <\kappa)$-generated", "$\mathcal{C}$ is weakly $(<\nu, <\kappa)$-generated" and their variants are self-explanatory (various intermediate senses of generation may also prove interesting).

The following theorem summarizes and supplements all our previous work.

22.1. Theorem. *Let Ω be a regular cardinal and $\aleph_\alpha \geqslant \Omega$. Let Q be a bounded p.o.s. in which each operation is $<\Omega$-ary. Then there is a bounded p.o.s. $\mathcal{C}$ of the same type, and there are sets W and G such that all the following hold*:

(1) *$\mathcal{C}$ is a substructure of Q^W, and contains the constant functions $W \to Q$, so that the natural embedding of Q in Q^W is a complete embedding of Q in $\mathcal{C}$.*

(2) *$\mathcal{C}$ is, moreover, a $<\Omega$-substructure of Q^W, hence, if $\nu \leqslant \Omega$ and is $<\nu$-complete, then Q is $<\nu$-complete.*

(3) *$G \subseteq C$, $|G| \leqslant \aleph_\alpha$ and G generates C by means of meets and joins existing in $\mathcal{C}$ (that is, G generates $\langle C; \leqslant^C \rangle$ in the $<\infty$-sense).*

(4) *If $|Q| \leqslant \aleph_\alpha$, then $W = 1$, $\mathcal{C} = Q^1$ (for this case the estimates in (5), (6) are uninteresting, for one can assert that $|C| = |Q|$ and G $(= C)$ generates $\mathcal{C}$ in the $<\aleph_0$-sense).*

(5) *Let $\kappa_1 = \min\{\kappa \mid |Q| \leqslant \aleph_\alpha^{<\kappa}, \Omega < \kappa\}$, $\kappa_2 = \min\{\kappa \mid \kappa_1 \leqslant \kappa, \Omega \leqslant \mathrm{cf}(\kappa)\}$ (thus $\kappa_1 \leqslant \kappa_2 \leqslant \kappa_1^+$ and if $|Q| \leqslant \aleph_\alpha^\Omega$, then $\kappa_1 = \kappa_2 = \Omega^+$). Then G generates $\mathcal{C}$ in the weak $<\kappa_1$-sense and in the $<\kappa_2$-sense (so $\mathcal{C}$ is $(\leqslant\aleph_\alpha, <\kappa_2)$-generated).*

(6) *Let ν_0 be the number of operation symbols in the type of Q. Then $|C| \leqslant (\max(|Q|, \nu_0))^{<\Omega}$ and $|C| = |Q|$ when $\Omega = \aleph_0$ and $\nu_0 \leqslant \max(\aleph_0, |Q|)$*

22.2. Remark. (1) In the proof (for the non-trivial case $|Q| > \aleph_0$) one should take $\mathcal{C}$ as the structure called $\mathcal{C}_0$ in §21.

(2) The reader is advised to put $\Omega = \aleph_0$ in the theorem (and possibly also $\aleph_\alpha = \aleph_0$) in order to see that it contains Theorem 16.1 (and 8.1) as a special case (clearly 22.1(1) implies that if $Q = \langle Q; \leqslant^Q, \wedge^Q, \vee^Q, \neg^Q, 1^Q, 0^Q \rangle$ is a B.a., then so is $\mathcal{C}$).

§23. Implications of the main result

The main points in Theorem 22.1 are clause (1) and the fact that $\mathcal{C}$ is $\leqslant\aleph_\alpha$-generated in various senses. Clause (2) can be subsumed under (1) in most of the interesting cases, where one incorporates the meet and join operations in the structure Q. Now, clause (1) implies that many interesting properties of Q are shared by $\mathcal{C}$, namely – all those properties which are preserved under direct powers and substructures.

The most useful sufficient condition for this is that the property be expressible by a universal Horn sentence. This is well-known for finitary languages, and is as easy to check in the general case, but let us explain briefly what we mean here. Let τ be the type of Q. From the operation symbols in τ and variables u_i, v_i (any i) we can form τ-terms (which are necessarily of length $<\Omega$). The relation symbols in τ, when applied to τ-terms, lead to τ-atomic-formulas, and among them equations and inequalities (since $\{\approx, \leqslant\} \subseteq \tau$). By the connectives and quantifiers $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$ we can form $\tau_{\infty\omega}$- or even $\tau_{\infty\infty}$-formulas (i.e., allow quantification on sets U_1, U_2, ... of bindable variables). The valuation of terms and satisfaction of formulas in assignments into structures of type τ is defined as usual. A universal Horn sentence is one of the form $(\forall U)$ $(\wedge X \to \Phi)$ where Φ is an atomic formula and X a set of atomic formulas. Just as for the finitary ($\mathcal{L}_{\omega\omega}$)-language, one proves that if σ is such a sentence, then $Q \models \sigma \Rightarrow Q^W \models \sigma \Rightarrow \mathcal{C} \models \sigma$.

In particular, every atomic formula that holds identically in Q holds identically in $\mathcal{C}$ (identically = for all assignments).

As a very special example take $\Omega = \aleph_0$ and suppose that $Q = \langle Q; \leqslant^Q, \wedge^Q, \vee^Q, 1^Q, 0^Q \rangle$ is a bounded modular lattice. Since modularity can be expressed by an equation, the theorem (with $\aleph_\alpha = \Omega = \aleph_0$) shows that Q has a complete embedding in a bounded modular lattice $\mathcal{C} = \langle C; \leqslant^C, ...\rangle$, such that $\mathcal{C}$ is countably generated (indeed $(\leqslant\aleph_0, <\kappa)$-generated where κ is the first infinite cardinal satisfying $|Q| \leqslant 2^{<\kappa}$) and $|C| = |Q|$. (Note that if $\wedge^Q$, $\vee^Q$ are the meet and join in $\leqslant^Q$, then the induced operations on C are the meet and join in $\leqslant^C$.) In this example "modular lattice" could be replaced by arbitrary lattice, distributive lattice etc. Considering lattices with additional operations, one gets the same results for Boolean algebras, lattices with (relative) pseudo-complementation, Stone algebras etc.

Next we consider examples with $\Omega > \aleph_0$. A poset is said to be $(<\nu_1, <\nu_2)$-distributive when the equality

$$\bigwedge_{i\in I} \bigvee_{j\in J_i} b_{ij} = \bigvee_{f\in \prod_{i\in I} J_i} \bigwedge_{i\in I} b_{if(i)}$$

holds in it for all families (b_{ij}) of elements for which $|I| < \nu_1$, $|J_i| < \nu_2$ for all $i \in I$, the left-hand side is defined and all meets on the right-hand side exist. $(\leqslant\nu_1, \leqslant\nu_2)$-distributivity is similarly defined.

Now suppose Q is a $<\Omega$-complete, $(<\nu_1, <\nu_2)$-distributive p.o.s. We may assume that the $<\Omega$-ary meet and join operations of $\langle Q; \leqslant^Q\rangle$ are incorporated in the structure $\mathcal{Q}$. Now, if $\nu_1, \nu_2 \leqslant \Omega$ and $[|I| < \nu_1$, $(\forall i \in I)$ $(|J_i| < \nu_2)$ imply $|\Pi_{i\in I} J_i| < \Omega]$, then the $(<\nu_1, <\nu_2)$-distributive law can be expressed by a set of equations involving the $<\Omega$-ary meet and join operations, and hence the law will hold in $\mathcal{C}$ as well. It is worth noting however, that the $(<\nu_1, <\nu_2)$-distributive law is preserved from $\mathcal{Q}$ to $\mathcal{C}$ whenever $\mathcal{Q}$ is $<\Omega$-complete and $\nu_1, \nu_2 \leqslant \Omega$, because for the index sets I, J_i $(i \in I)$ satisfying $|I| < \Omega$, $|J_i| < \Omega$ $(i \in I)$ it can be expressed by the universal Horn sentence

$$(\forall u_0)\,(\forall \{u_{ij} \mid i \in I, j \in J_i\})\left[\bigwedge\!\!\!\bigwedge_{f\in \prod_{i\in I} J_i} \left(\bigwedge_{i\in I} u_{if(i)} \leqslant u_0\right) \rightarrow \left(\bigwedge_{i\in I} \bigvee_{j\in J_i} u_{ij} \leqslant u_0\right)\right]$$

where $\wedge$, $\vee$ are various $<\Omega$-ary operation symbols for meet and join (all assumed to belong to the type τ of $\mathcal{Q}$), while $\bigwedge\!\!\!\bigwedge$ is the conjunction in the language $\tau_{\infty\infty}$. This proves the following result.

23.1. Theorem. *If in 22.1 $\mathcal{Q}$ is $<\Omega$-complete and $(<\nu_1, <\nu_2)$-distributive, and $\nu_1, \nu_2 \leqslant \Omega$, then $\mathcal{C}$ is ($<\Omega$-complete and) $(<\nu_1, <\nu_2)$-distributive.*

The reasoning leading to 23.1 breaks down when the $(<\Omega, <\mu)$-distributive law is considered for $\mu > \Omega$. For the special case of B.a's we shall prove by a forcing argument that if $\mathcal{Q}$ is $(<\Omega, <\infty)$-distributive so is $\mathcal{C}$, but I do not know if this holds for lattices and what happens if $\Omega < \mu < \infty$.

§24. Classes of partially ordered structures

So far we have been considering a single structure $\mathcal{Q}$, trying to embed it in a $\leqslant\Omega$-(or $\leqslant\aleph_\alpha$-)generated structure of the same type. For B.a's, however, we have had also translation results like 12.8, and results about embeddings and cardinalities of free algebras. We may expect that such results can be generalized if the class of B.a's is replaced by a class Λ of p.o.s's satisfying some conditions. We shall now sketch how this can be done, but we shall not even try to generalize the results pertaining to complete B.a's and those dealing with the number of disjoint elements, because the proofs of these use rather specific properties of B.a's.

Suppose we are given a regular cardinal Ω, a type τ of p.o.s's such that each operation symbol of τ is $<\Omega$-ary, and a class Λ of bounded p.o.s's of type τ (think of the case $\Omega = \aleph_0$, τ the type of B.a's and Λ the class of B.a's). First we wish to define the lattice-τ-terms, which are the analogues of Boolean terms. These are formed from the variables (which we prefer to denote now p_i rather than u_i, v_i) just like the τ-terms of §23 except that we allow two additional operations $\bigwedge$ and $\bigvee$, which may be applied to any set of terms and which are interpreted in valuations as meet and join (a valuation is a pair (Q, I) where Q is a p.o.s. of type τ and I a function into Q). As in the case of B.t's we allow an improper value * to handle the "undefined" case. Note that if the type τ of B.a's contains, say, besides $\approx$ and $\leqslant$ also two binary operations-symbols for meet and join, a unary operation symbol for complementation, and constants for 1 and 0, then the lattice-τ-terms are not exactly the B.t's defined earlier, but the differences are inessential. From the lattice-τ-terms we may form atomic formulas by the relation symbols in τ. We may also form (lattice-τ)$_{\infty\infty}$-formulas, but these are not needed here.

We denote lattice-τ-terms by $\phi, \chi, \psi, \ldots$ and atomic formulas by $\Phi, \Psi, \ldots$. A (lattice-τ-)atomic formula $\Phi = R_i^\eta(\langle \phi_j \mid j \in \eta \rangle)$ is said to be defined in the valuation (Q, I) when each ϕ_j is defined, and then Φ is true (satisfied) or false according as $\langle \|\phi_j\|_{Q,I} \mid j \in \eta \rangle$ is a member of $(R_i^\eta)^Q$ or not.

So far the definitions have referred only to τ. The class Λ is important for the following definition: Let Γ be a set of atomic formulas, Φ an atomic formula. We write $\Gamma \vdash \Phi$ when for every valuation (Q, I), if $Q \in \Lambda$ and in (Q, I) all members of Γ are true and Φ is defined, then Φ is true. $\vdash \Phi$ means $\emptyset \vdash \Phi$ (i.e., Φ is Λ-valid).

We now define:

$\phi \equiv \psi(\Gamma)$ when $\Gamma \vdash \phi \approx \psi$ (equivalently, $\Gamma \vdash \phi \leqslant \psi$ and $\Gamma \vdash \psi \leqslant \phi$).
$\phi \equiv \psi$ when $\phi \equiv \psi(\emptyset)$.

Surprisingly enough, it is not obvious at all that $\equiv(\Gamma)$ or even $\equiv$ is a transitive relation (I suspect this is false for some choices of Λ but have not checked it) because if $\phi \equiv \chi$ and $\chi \equiv \psi$ and ϕ, ψ are defined in (Q, I)

then χ need not be defined and it is conceivable that $\|\phi\| \neq \|\psi\|$ in (Q, I). Therefore we introduce the following assumption on Λ.

(I) For every $Q \in \Lambda$ and every infinite cardinal κ there is a $<\kappa$-complete $Q' \in \Lambda$ and a $<\kappa$-complete embedding of Q in Q'.

(In the case of B.a's one takes Q' as the normal completion of Q.) Assuming (I), $\equiv(\Gamma)$ is easily seen to be an equivalence relation and moreover, if $\langle \phi_i \mid i \in \eta \rangle$, $\langle \psi_i \mid i \in \eta \rangle$ are any two families of lattice-τ-terms such that $\phi_i \equiv \psi_i(\Gamma)$ for all $i \in \eta$, then

$$\bigwedge_i \phi_i \equiv \bigwedge_i \psi_i(\Gamma), \qquad \bigvee_i \phi_i \equiv \bigvee_i \psi_i(\Gamma),$$

$$\langle \phi_i \mid i \in \eta \rangle \equiv o \, \langle \psi_i \mid i \in \eta \rangle \, (\Gamma)$$

whenever $o = O_j^\eta$ is an operation symbol in τ, and $\Gamma \vdash r\langle \phi_i \mid i \in \eta \rangle$ iff $\Gamma \vdash r\langle \psi_i \mid i \in \eta \rangle$ whenever $r = R_j^\eta$ is a relation symbol in τ.

In order to apply Theorem 22.1 in the class Λ we assume further:

(II) Λ is closed under direct products and $<\Omega$-substructures.

We can now repeat the work of § 12. Let q_{MN}, $\bar{q}_{MN}$, Γ_N $(M, N < \Omega)$ be distinct individual variables and define the lattice-terms π_x^N by recursion on x (cf. 12.3):

$$\pi_x^N = \bigwedge_M \Big(\bar{q}_{MN} \vee \bigvee_{y \in x} \pi_y^M \Big) \wedge \bigwedge_{y \in x} \bigvee_M (q_{MN} \wedge \pi_y^M)$$

$(M, N$ vary on Ω.) Then let $\rho_x^0 = \bigvee_N (\pi_x^N \wedge r_N)$, and define a translation K of arbitrary lattice-τ-terms into those on the variables q_{MN}, $\bar{q}_{MN}$, r_N by:

$K(p_x) = \rho_x^0$ and K commutes with $\wedge$, $\vee$ and the operation symbols in τ. Extend K in the obvious way to atomic formulas and sets of them. Then the considerations of § 12 can be adapted to get the following analogue of Theorem 12.8 (after proving a "rule of substitution" for $\vdash$ and noting that, by (II), $Q \in \Lambda$ implies that $\mathcal{C} \in \Lambda$ where $\mathcal{C}$ is the p.o.s. of §21).

24.1. Theorem. *For any set* $\Gamma \cup \{\Phi\}$ *of lattice-τ-atomic formulas,* $\Gamma \vdash \Phi$ *iff* $K(\Gamma) \vdash K(\Phi)$.

Next we note that assumption (II) enables us to construct a free $<\kappa$-complete p.o.s. over Λ on any number of generators where κ is any regular cardinal $\geqslant \Omega$. One can take all lattice-τ-terms on the variables p_i,

$i \in A$ (any set A) of length $<\kappa$, divide them by the congruence relation $\equiv$ and define the operations and relations on the equivalence classes by taking representatives (the classes $[\phi_i]$, $i \in \eta$ stand in the relation corresponding to R_j^η ($\in \tau$) iff $\vdash R_j^\eta \langle \phi_i \mid i \in \eta \rangle$). This gives a $<\kappa$-complete p.o.s. $\mathcal{F}^{<\kappa}(A)$ of type τ, which is $<\kappa$-generated by the equivalence classes $[p_i]$, $i \in A$, and whenever $Q \in \Lambda$ is $<\kappa$-complete, $I : A \to Q$, there is a unique $<\kappa$-complete homomorphism $h : \mathcal{F}^{<\kappa}(A) \to Q$ such that $h([p_i]) = I(i)$ for all $i \in A$. The interesting point is that $\mathcal{F}^{<\kappa}(A)$ is isomorphic to an element of Λ. This follows from assumption (II) by adapting a well-known device due to Birkhoff for constructing a free algebra in terms of direct products and subalgebras (cf. [4, §25, Corollary 1]). One reason for taking $\kappa \geqslant \Omega$ is that we are only assuming closure of Λ under $<\Omega$-substructures and in adapting Birkhoff's proof we take the product of many $<\kappa$-substructures of elements of Λ. We could also consider $\mathcal{F}^{<\kappa}(A)$ for certain singular cardinals κ, but this can be left to the interested reader. If (II) is strengthened to closure under arbitrary substructures (and direct products) things become even simpler.

We can now repeat the argument at the beginning of § 14 using 24.1 instead of 12.8 to get (cf. 14.1):

24.2. Theorem. *Let κ be a regular cardinal $>\Omega$, $\nu \leqslant 2^{<\kappa}$. Then there is a $<\kappa$-complete embedding of $\mathcal{F}^{<\kappa}(\nu)$ in $\mathcal{F}^{<\kappa}(\Omega)$.*

(When $2^{<\Omega} = \Omega$, as is the case for $\Omega = \aleph_0$, the theorem is true also for $\kappa = \Omega$ because we have $\nu \leqslant \Omega$. But when $2^{<\Omega} > \Omega$ it is not clear whether 24.2 holds for $\kappa = \Omega$.)

Analogues of 24.1 and 24.2 can also be given in the more general setting of $\leqslant\aleph_\alpha$ generators as in §§ 15–16, but we have no space for details.

Finally the reader might wish to know whether we can determine the exact power of $\mathcal{F}^{<\kappa}(\Omega)$ (or $\mathcal{F}^{<\kappa}(\aleph_\alpha)$), and on the whole whether our cardinality estimates are best possible in the same strong sense that they are for B.a's. It turns out that the following further assumptions suffice to ensure that Λ behaves exactly like the class of B.a's in all these respects (so that, e.g. $\mathcal{F}^{<\kappa}(\aleph_\alpha) = \aleph_\alpha^{<\kappa}$ etc.):

(III) Λ contains at least one structure with more than one element.

(IV) $\Omega = \aleph_0$ and $\nu_0 \leqslant \aleph_0$ where ν_0 is the number of operation symbols in τ.

Thus, if Ω, τ, Λ satisfy (besides the initial assumptions) (I)–(IV) then all our results about the class of B.a's, except possibly the aspects concerned with complete B.a's or with the number of disjoint elements, are good for the class Λ. The reader will have no difficulty in verifying this, once he notes that (III) implies that the generators $[p_i]$, $i \in A$ of $\mathcal{F}^{<\kappa}(A)$ are pairwise distinct, and hence (using (II), (IV)) that Λ contains structures of any infinite power.

Unfortunately, the only interesting cases I know in which (I)–(IV) hold, besides the cases of arbitrary bounded lattices and B.a's, are the class of lattices $\langle \mathcal{L}; \wedge, \vee, {}^*, 1, 0\rangle$ with pseudo-complementation, the class of lattices $\langle \mathcal{L}; \wedge, \vee, {}^*, 1, 0\rangle$ with relative pseudo-complementation [5, pp. 58, 61, Ex. 25] and the corresponding classes of meet-semi-lattices. These classes are equational [5, Ex. 23, 26] and are known to be closed under completion by cuts and so satisfy assumption (I). (See also [18, Ch. IV, 1.1 and 9.1].)

For many natural classes, e.g. the class of bounded modular lattices, assumption (II) is satisfied ($\Omega = \aleph_0$) and hence the theorem of §22 is useful, but it has not yet been checked how much of the work on B.a's carries over, and what new distinctions or concepts must be introduced in order to clarify this.

§25. The generalized Gaifman–Hales theorem

Since Kripke's embedding theorem entails the Gaifman–Hales theorem, we have concentrated in §§19–24 on extending the former. If we apply the same ideas to the proof of the Gaifman–Hales theorem in §1, we arrive at the following observations:

Let Ω be a regular cardinal, A a transitive non-empty set. Consider the two-valued model $M = \langle A, \in\rangle$ for the language $\mathcal{L} = \{\epsilon\}$, and define the locating formulas as in §2. Put $W = \{x \mid x : \Omega \to A\}$ (the space of assignments into M), and call a subset X of W cylindric when there is an $N < \Omega$ such that, for $x, y \in W$, $x \restriction N = y \restriction N$ implies ($x \in X$ iff $y \in X$) (this is the natural specialization to $Q = \{0, 1\}$ of the definition of cylindric members of Q^W in §20). The collection C of cylindric subsets of W is a $<\Omega$-complete field of sets, and hence determines a $<\Omega$-complete B.a. $\mathcal{C}$. Members of C can be represented in the form

$\widetilde{\phi} = \{x \in W \mid M \models \phi[x]\}$, for $\mathcal{L}_{\infty\omega}(M)$ formulas ϕ, whose set of free variables is a subset of $\{v_N \mid N < \Omega\}$ of power $<\Omega$ (so called "admissible formulas").

In particular, putting

$$T_0 = \bigcup \{\mathrm{Sub}^*(\pi_x(v_N)) \mid x \in A,\ N < \Omega\},$$

$$T_1 = T_0 \cup \left\{ \bigwedge_{N<N_0} \pi_{p(N)}(v_N) \mid N_0 < \Omega,\ p : N_0 \to A \right\},$$

$$T_2 = T_1 \cup \{\bigvee X \mid X \subseteq T_1,\ \bigvee X \text{ is admissible}\},$$

we have $\widetilde{T}_0 \subseteq \widetilde{T}_1 \subseteq \widetilde{T}_2 = C$, and we can show that $\widetilde{T}_1$ is a dense subset of $\mathcal{C}$ and $\mathcal{C}$ is generated by $\{\widetilde{\phi} \mid \phi \in T_0,\ \phi \text{ atomic}\}$, which is of power $\leqslant\Omega$.

When $\Omega = \aleph_0$ we have the Gaifman–Hales theorem. What is gained by considering an arbitrary regular Ω? The main point is that $\mathcal{C}$ is $(<\Omega, <\infty)$-distributive. We could prove this directly, but it is simpler to relate $\mathcal{C}$ to B.a's discussed in the literature. Consider W as a topological space with the basis $\{B_p \mid p \in P\}$ where $P = \{p \mid (\exists N < \Omega)\,(p : N \to A)\}$ and for $p \in P$, $B_p = \{x \in W \mid p \subseteq x\}$. It is clear that each cylindric set is open and closed, and that each set B_p $(p \in P)$ is cylindric. Therefore $\mathcal{C}$ (and any dense subalgebra of $\mathcal{C}$) is a dense subalgebra of $\mathrm{RO}(W)$ (= the complete B.a. of regular open sets of the space W), so that $\mathrm{RO}(W)$ is isomorphic to the normal completion of $\mathcal{C}$ (hence $\mathrm{RO}(W)$ is $\leqslant\Omega$-generated too). To show that $\mathcal{C}$ is $(<\Omega, <\infty)$-distributive, it suffices to show the same for $\mathrm{RO}(W)$, and this is known from [19].

Solovay used (in [22]) B.a's of the form $\mathrm{RO}(W)$ to prove the Gaifman–Hales theorem in the general form:

25.1. Theorem [3, 7]. *Let Ω be a regular cardinal. Then there exist arbitrarily large complete $(<\Omega, <\infty)$-distributive $\leqslant\Omega$-generated B.a's.*

Here we see that the fact that RO(W) is $\leqslant\Omega$-generated follows from the representation of cylindric sets by formulas in a natural way. Moreover, the fact that we are using assignments into an arbitrary transitive non-empty set A rather than into an ordinal only enables us, by considering suitable subalgebras of $\mathcal{C}$ (e.g. $[\widetilde{T}_1]^{<\aleph_0}$) to prove results like the following:

If Ω is regular $\kappa > \Omega$ and $\mathrm{cf}(\kappa) \geq \Omega$, then the maximum power of a $(\leq\Omega, <\kappa)$-generated, $(<\Omega, <\infty)$-distributive B.a. is exactly $2^{<\kappa}$ $(= \Omega^{<\kappa})$.

We shall not go into further cardinality estimates of this kind, nor into the generalizations to $\leq\aleph_\alpha$-generated $(<\Omega, <\infty)$-distributive B.a.'s, mainly to save space and time.

Although Solovay gave a computational proof that the complete B.a.'s $\mathcal{B}$ which he considered are countably (or $\leq\Omega$-) generated, he had actually arrived at the result by considering the Boolean-valued models $V^{(\mathcal{B})}$ of set theory (see [20]), which are intimately connected with Cohen's notion of forcing. A "conceptual" proof that $\mathrm{RO}(W)$ is $\leq\Omega$-generated and $(<\Omega, <\infty)$-distributive using basic facts about forcing and Scott–Solovay models of set theory can easily be extracted from some portions of the proof in §26, where an extension of Kripke's theorem is derived by these methods. Though unpublished, such proofs are probably well-known. From now on we have to assume that the reader is acquainted with the method of forcing. The most convenient source for our purposes is Jech's lectures [8], especially §§16–19. We shall use the terminology, notation and results of §§16–17, and follow Jech in taking a countable transitive $\in$-model $\mathfrak{M}$ as the ground model, though we could start with the universe V as in [20]. For more details why one may assume that $\mathfrak{M}$ satisfies each axiom of ZFC (separately) and why one may prove theorems in ZFC by showing that they hold in $\mathfrak{M}$ see [14, pp. 132–133].

Our only deviation from Jech's terminology is that in his definition of an $\mathfrak{M}$-generic set of conditions [8, p. 52] we replace condition (b) by:

$$(\forall x, y \in G)\,(\exists z \in G)\,(z \leq x \;\&\; z \leq y)\,,$$

and call G an $\mathfrak{M}$-generic *filter* over $\mathcal{P}$. [8, Lemma 45] is true for the modified definition.

Given a B.a. $\mathcal{B}$ and a poset P, we say that P is dense in $\mathcal{B} \sim \{0\}$ when P is a subposet of $\mathcal{B}$, not containing 0, and $(\forall b \in \mathcal{B} \sim \{0\})\,(\exists p \in P)$ $(p \leq b)$.

We shall make extensive use of the following facts [8, p. 49, Theorem 29B and p. 52, Lemma 45]:

(1) If $\mathcal{B}$ is a B.a., then every dense subposet of $\mathcal{B} \sim \{0\}$ is separative (as defined in [8, p. 48]).

(2) If P is a separative poset then there is a complete B.a. $\mathcal{B}$, unique

up to isomorphism over P (and denoted by RO(P)) such that P is a dense subposet of $\mathcal{B} \sim \{0\}$.

(3) If in the model $\mathfrak{M}$, $\mathcal{B}$ is a complete B.a. and P a dense subposet of $\mathcal{B} \sim \{0\}$, then there is a 1–1 correspondence between $\mathfrak{M}$-generic ultrafilters (u.f's) G on $\mathcal{B}$ and $\mathfrak{M}$-generic filters G_1 on P given by

$$G_1 = G \cap P,$$

$$G = \{b \in \mathcal{B} \mid (\exists p \in G_1)\, p \leqslant b\}$$

(and thus $\mathfrak{M}[G] = \mathfrak{M}[G_1]$).

The reader is also advised to read about cardinal collapsing [8, § 18, Model V, pp. 70–71] and the proof of Kripke's theorem [8, § 19]. A proof of this kind was first found by Kripke, and its ideas can be used to get other results about B.a's [8, § 19, pp. 76–78], though apparently not the results of the present work.

§ 26. A generalization of Kripke's theorem to $(<\Omega, <\infty)$-distributive B.a's

26.1. Theorem. *Let Ω be a regular cardinal and $\mathcal{B}_0$ a $(<\Omega, <\infty)$-distributive B.a. Then $\mathcal{B}_0$ has a complete embedding in some complete, $(<\Omega, <\infty)$-distributive $\leqslant\Omega$-generated B.a. $\mathcal{C}'$.*

After proving this by a forcing argument we shall connect $\mathcal{C}'$ with the B.a. $\mathcal{C}$ of cylindric functions obtained from $\mathcal{B}_0$ by the method of § 21. Since that construction depended on a representation of $\mathcal{B}_0$ (there called Q) as an indexed set we shall start here also from a representation $\mathcal{B}_0 = \{b_i \mid i \in A\}$, A transitive, and construct $\mathcal{C}'$ from it (though for proving 26.1 in itself A could be assumed an ordinal).

26.2. Lemma. *Let $\mathcal{B}$ and $\mathcal{D}$ be non-degenerate B.a's, and let P_1 and P_2 be dense subposets of $\mathcal{B} \sim \{0\}$, $\mathcal{D} \sim \{0\}$, resp. Then there is a complete B.a. $\mathcal{C}'$ such that $P_1 \times P_2$ is a dense subposet of $\mathcal{C}' \sim \{0\}$, and $\mathcal{C}'$ is unique up to isomorphism for given $\mathcal{B}$ and $\mathcal{D}$ (independently of the choice of P_1, P_2). Moreover, $\mathcal{B}$ as well as $\mathcal{D}$, has a complete embedding in $\mathcal{C}'$.*

Proof. *Existence*: Since P_1 and P_2 are separative so is $P_1 \times P_2$, hence $P_1 \times P_2$ is a dense subposet of $\mathcal{C}' \sim \{0\}$ for some complete B.a. $\mathcal{C}'$.

Uniqueness: Let $\mathcal{C}'_0$ be the unique (up to isomorphism) complete B.a. such that $(\mathcal{B} \sim \{0\}) \times (\mathcal{D} \sim \{0\})$ is a dense subposet of $\mathcal{C}'_0 \sim \{0\}$. Then, if P_1 is dense in $\mathcal{B} \sim \{0\}$, P_2 in $\mathcal{D} \sim \{0\}$ then $P_1 \times P_2$ is dense in $\mathcal{C}'_0 \sim \{0\}$, and thus $\mathcal{C}'_0$ is good for any choice of P_1, P_2.

Embedding: Without loss of generality, $P_1 = \mathcal{B} \sim \{0\}$, $P_2 = \mathcal{D} \sim \{0\}$. Since $\mathcal{D}$ is non-degenerate $1^{\mathcal{D}} \in P_2$. Define $h : \mathcal{B} \to \mathcal{C}'$ by $h(0) = 0$ and $h(b) = (b, 1)$ for $b > 0$. It is straightforward to check (using the density of $P_1 \times P_2$ is $\mathcal{C}' \sim \{0\}$) that h preserves complements and (infinite) meets, hence is a complete embedding of $\mathcal{B}$ in $\mathcal{C}'$. $\mathcal{D}$ is embedded in $\mathcal{C}'$ similarly. $\square$

26.3. Remark. The B.a. $\mathcal{C}'$ is actually well-known. It is the normal completion of the "tensor product" $\mathcal{B} \otimes \mathcal{D}$ and one way to construct it explicitly (see [24, 5.17–1.18] is this: Let X_1, X_2 be the Stone spaces of $\mathcal{B}$, $\mathcal{D}$, resp., and let $\mathcal{C}'$ be (isomorphic to) the regular-open algebra of $X_1 \times X_2$. This may be used to give an alternative proof of the lemma (except possibly for the uniqueness).

Theorem 26.1 is an immediate corollary of Lemma 26.2 and the following result.

26.4. Theorem. *Let Ω be a regular cardinal, $\mathcal{B}_0$ a non-degenerate $(<\Omega, <\infty)$-distributive B.a., $\mathcal{B}_0 = \{b_i \mid i \in A\}$, A transitive, P_1 a dense subposet of $\mathcal{B}_0 \sim \{0\}$. Let $P_2 = \{p \mid (N < \Omega), p : N \to A\}$, with the partial order $p \leqslant q$ iff $q \subseteq p$.*

Let $\mathcal{C}'$ be a complete B.a. such that $P_1 \times P_2$ is a dense subposet of $\mathcal{C}' \sim \{0\}$. Then

(I) *$\mathcal{C}'$ is $(<\Omega, <\infty)$-distributive;*

(II) *$\mathcal{C}'$ is $\leqslant\Omega$-generated.*

26.5. Note. P_1 plays no role in the theorem, and could be chosen as $\mathcal{B}_0 \sim \{0\}$, but by Lemma 26.2 the choice of P_1 does not affect $\mathcal{C}'$. The existence of $\mathcal{C}'$ follows easily once one notes that: if $\mathcal{B}_0$ is non-degenerate, then $|A| > 1$ which implies that P_2 is separative. By Lemma 26.2 $\mathcal{B}_0$ has a complete embedding in $\mathcal{C}'$, and this proves 26.1 (the case of degenerate $\mathcal{B}_0$ being trivial).

To prove 26.4 we shall use three lemmas known from the literature.

26.6. Lemma (Pierce). *Let $\mathcal{B}_0$ be a $(<\Omega, <\infty)$-distributive B.a., $\mathcal{B}_1$ its normal completion. Then $\mathcal{B}_1$ is $(<\Omega, <\infty)$-distributive.*

This is [17, Corollary 4.7]. The other two lemmas are concerned with a fixed countable transitive $\in$-model $\mathfrak{M}$ which satisfies each axiom of ZFC. For each ordinal α we put $\mathfrak{M}^{<\alpha} = \{f \mid (\exists\beta < \alpha)\ f : \beta \to \mathfrak{M}\}$.

26.7. Lemma (Scott). *In $\mathfrak{M}$ let Ω be a regular cardinal, $\mathcal{B}$ a complete B.a. Then $\mathcal{B}$ is $(<\Omega, <\infty)$-distributive in $\mathfrak{M}$ iff for each $\mathfrak{M}$-generic u.f. G on B, $\mathfrak{M}[G] \cap \mathfrak{M}^{<\Omega} = \mathfrak{M} \cap \mathfrak{M}^{<\Omega}$ (i.e. $\mathfrak{M}[G]$ does not contain any new sequences of length $<\Omega$ of elements of $\mathfrak{M}$).*

Though not explicitly stated in this way, the theorem is essentially proved in [20, pp. 56–60].

26.8. Lemma (Solovay). *Let P_1, P_2 be posets with 1 (i.e., each has a greatest element) in $\mathfrak{M}$, and let G be an $\mathfrak{M}$-generic filter on $P_1 \times P_2$. Put*

$$G_1 = \{p_1 \in P_1 \mid (p_1, 1) \in G\}\ ,$$

$$G_2 = \{p_2 \in P_2 \mid (1, p_2) \in G\}\ .$$

Then $G = G_1 \times G_2$, G_1 is an $\mathfrak{M}$-generic filter on P_1, G_2 is an $\mathfrak{M}[G_1]$-generic filter on P_2 and $\mathfrak{M}[G] = (\mathfrak{M}[G_1])[G_2]$.

This is a part of the Product Theorem [2, §8], which is due to Solovay (cf. [23, 2.3]).

We are now ready for the proof of 26.4. It is enough to prove that the theorem holds in the model $\mathfrak{M}$. So let Ω, $\mathcal{B}_0$, A, $\langle b_i \mid i \in A\rangle$, P_1, P_2 and $\mathcal{C}'$ be elements of $\mathfrak{M}$ which satisfy all the assumptions in $\mathfrak{M}$. We shall prove that assertions (I) and (II) about $\mathcal{C}'$ hold in $\mathfrak{M}$.

Proof of (I). By Lemma 26.7 it suffices to show that if H is any $\mathfrak{M}$-generic u.f. on $\mathcal{C}'$, then $\mathfrak{M}[H] \cap \mathfrak{M}^{<\Omega} = \mathfrak{M} \cap \mathfrak{M}^{<\Omega}$. Let H be such an u.f. and put $G = H \cap (P_1 \times P_2)$. Then G is an $\mathfrak{M}$-generic filter on $P_1 \times P_2$. Without loss of generality $1^{\mathcal{B}_0} \in P_1$. P_2 also has a greatest element. Let G_1 and G_2 be the filters on P_1, P_2 formed from G in Lemma 26.8.

Since $\mathfrak{M}[H] = \mathfrak{M}[G] = (\mathfrak{M}[G_1])[G_2]$ it suffices to show that:
(a) $\mathfrak{M}[G_1] \cap \mathfrak{M}^{<\Omega} = \mathfrak{M} \cap \mathfrak{M}^{<\Omega}$;
(b) $(\mathfrak{M}[G_1])[G_2] \cap \mathfrak{M}^{<\Omega} = \mathfrak{M}[G_1] \cap \mathfrak{M}^{<\Omega}$.

Proof of (a): Let $\mathcal{B}_1$ be the normal completion of $\mathcal{B}_0$ in $\mathfrak{M}$. By Lemma 26.6 $\mathcal{B}_1$ is $(<\Omega, <\infty)$-distributive (in $\mathfrak{M}$). P_1 is a dense subposet of $\mathcal{B}_1 \sim \{0\}$. Since G_1 is $\mathfrak{M}$-generic on P_1, there is an $\mathfrak{M}$-generic u.f. G_1' on B_1 such that $\mathfrak{M}[G_1] = \mathfrak{M}[G_1']$. By Lemma 2.67, $\mathfrak{M}[G_1'] \cap \mathfrak{M}^{<\Omega} = \mathfrak{M} \cap \mathfrak{M}^{<\Omega}$, hence (a).

Proof of (b): In $\mathfrak{M}$, P_2 has (by the regularity of Ω) the following property:

each descending sequence (in P_2) of length $<\Omega$ has a lower bound.

It follows from (a) that P_2 has the same property in $\mathfrak{M}[G_1]$. By [8, p. 66, Lemma 57] it follows (since G_2 is an $\mathfrak{M}[G_1]$-generic filter on P_2) that $(\mathfrak{M}[G_1])[G_2] \cap (\mathfrak{M}[G_1])^{<\Omega} = \mathfrak{M}[G_1] \cap (\mathfrak{M}[G_1])^{<\Omega}$, and this clearly implies (b) and completes the proof of (I).

Proof of (II). We first explain the intuition behind the proof: Let H be an $\mathfrak{M}$-generic u.f. on $\mathcal{C}'$. Put $G = H \cap (P_1 \times P_2)$, $G_1 = \{p_1 \in P_1 \mid (p_1, 1) \in G\}$, $G_2 = \{p_2 \in P_2 \mid (1, p_2) \in G\}$ (again we assume that $1^{\mathcal{B}_0} \in P_1$). Then, as we know, G_1 and G_2 are $\mathfrak{M}$-generic filters on P_1, P_2 resp. It follows easily that $F = \bigcup G_2$ is a function from Ω onto A (P_2 is the poset one would naturally use to "collapse" $|A|$ to Ω). The relation $R = \{(M,N) \mid M,N < \Omega, F(M) \in F(N)\}$ can be used to "code" F, and indeed since A is a transitive set one may reconstruct F from R by noting that for any $N < \Omega$, $x \in A$:

$$\begin{aligned} F(N) = x \iff x &= \{F(M) \mid R(M,N)\} \\ &\iff \{(\forall M)\,[R(M,N) \Rightarrow (\exists y \in x)\, F(M) = y], \\ &\qquad \& \ (\forall y \in x)(\exists M)\,[R(M,N) \;\&\; F(M) = y]\} \end{aligned}$$

(thus $\{N \mid F(N) = x\}$ is determined from R by $\in$-recursion on x). (Note the similarity with Definition 2.1 of the locating formulas.) Since $G_2 = \{F \mid N \mid N < \Omega\}$, we see that the information needed to specify G_2 is contained in R ($\subseteq \Omega \times \Omega$). What about G_1? Since

$$G_1 \subseteq P_1 \subseteq \mathcal{B}_0 = \{b_i \mid i \in A\} = \{b_{F(N)} \mid N < \Omega\},$$

it is clear that G_1 is determined from F (hence R) and $S = \{N < \Omega \mid b_{F(N)} \in G_1\}$. Thus both G_1 and G_2, hence $G = G_1 \times G_2$, hence H, are coded by R, S where $R \subseteq \Omega \times \Omega, S \subseteq \Omega$.

We have so far been working with a particular $\mathfrak{M}$-generic u.f. H on $\mathcal{C}'$. If we consider instead of H, G, G_1, G_2, F, R and S the elements $\boldsymbol{H}, \boldsymbol{G}, \ldots, \boldsymbol{S}$ of $\mathfrak{M}^{\mathcal{C}'}$ which denote $H, G_1, \ldots, S$ under every choice of H, then for every $c \in \mathcal{C}'$ we have $c = \|\check{c} \in \boldsymbol{H}\|$ and it is reasonable by the above to suppose that $\|\check{c} \in \boldsymbol{H}\|$ can be obtained by $\neg^{\mathcal{C}'}, \wedge^{\mathcal{C}'}, \vee^{\mathcal{C}'}$ from the elements $\|(M,N)\check{} \in \boldsymbol{R}\|$, $\|\check{N} \in \boldsymbol{S}\|$ $(M,N < \Omega)$ of $\mathcal{C}'$, so that these $\leqslant\Omega$ elements generate $\mathcal{C}'$.

The precise proof is very simple now, but we urge the reader to recall [8, pp. 53–58] (especially Lemma 50 (p. 57), the notations of p.58 and Theorem 32), because they will be used repeatedly and without mention. Our basic B.a. is here $\mathcal{C}'$, and we let $\boldsymbol{H}$ (rather than $\boldsymbol{G}$) denote the canonical generic u.f. in $\mathfrak{M}^{\mathcal{C}'}$. "$\boldsymbol{H}$" varies on $\mathfrak{M}$-generic u.f.'s on $\mathcal{C}'$, so that for every H, $i_H(\boldsymbol{H}) = H$. We shall also use the fact that if $c_1, c_2 \in \mathcal{C}'$ and ($c_1 \in H$ iff $c_2 \in H$) for all H, then $c_1 = c_2$ (this follows from the countability of $\mathfrak{M}$). We shall write "$\mathfrak{M}^{\mathcal{C}'} \models \Phi$" for "$\|\Phi\| = 1$ in $\mathfrak{M}^{\mathcal{C}'}$", and say that Φ holds in $\mathfrak{M}^{\mathcal{C}'}$ (Φ is any set theoretical statement with parameters from $\mathfrak{M}^{\mathcal{C}'}$).

Proceeding to the proof itself, choose $\boldsymbol{G}, \boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{F}, \boldsymbol{R}, \boldsymbol{S} \in \mathfrak{M}^{\mathcal{C}'}$ such that the following hold in $\mathfrak{M}^{\mathcal{C}'}$ (we assume $1^{\mathcal{B}_0} \in P_1$):

$$\boldsymbol{G} = \boldsymbol{H} \cap (P_1 \times P_2),$$

$$\boldsymbol{G}_1 = \{p_1 \in \check{P}_1 \mid (p_1, 1) \in \boldsymbol{G}\},$$

$$\boldsymbol{G}_2 = \{p_2 \in \check{P}_2 \mid (1, p_2) \in \boldsymbol{G}\},$$

$$\boldsymbol{F} = \bigcup \boldsymbol{G}_2,$$

$$\boldsymbol{R} = \{(M,N) \mid M,N < \check{\Omega}, \boldsymbol{F}(M) \in \boldsymbol{F}(N)\},$$

$$\boldsymbol{S} = \{N < \check{\Omega} \mid \check{\bar{b}}(\boldsymbol{F}(N)) \in \boldsymbol{G}_1\},$$

where $\bar{b} = \langle b_i \mid i \in A\rangle$.

It is clear that for any H, if $G, \ldots, S$ are defined as above then $G = i_H(\boldsymbol{G}), \ldots, S = i_H(\boldsymbol{S})$. We now consider the following elements of $\mathcal{C}'$:

$$r_{MN} = \|(M,N)\check{} \in \boldsymbol{R}\| \quad (M,N < \Omega),$$

$$s_N = \|\check{N} \in \boldsymbol{S}\| \quad (N < \Omega)$$

$$f_{Nx} = \|\boldsymbol{F}(\check{N}) = \check{x}\| \quad (N < \Omega, x \in A).$$

Let $\mathcal{C}_1$ be the $<\infty$-subalgebra of $\mathcal{C}'$ generated by $\{r_{MN} \mid M, N < \Omega\}$, and let $\mathcal{C}_2$ be the $<\infty$-subalgebra of $\mathcal{C}'$ generated by $\{r_{MN} \mid M, N < \Omega\} \cup \{s_N \mid N < \Omega\}$ (more precisely $\mathcal{C}'$ and the families $\langle r_{MN} \mid M, N < \Omega\rangle$, $\langle s_N \mid N < \Omega\rangle$ are in $\mathcal{C}$, and $\mathcal{C}_1$, $\mathcal{C}_2$ are specified by the above definitions in $\mathcal{M}$). By proving that $\mathcal{C}_2 = \mathcal{C}'$ we shall complete the proof of (II).

First we prove that $f_{Nx} \in \mathcal{C}_1$ for all $N < \Omega$, $x \in A$. We have seen before how to reconstruct F from R, and since in that discussion H was arbitrary it is clear that for any $x \in A$, $N < \Omega$:

$$f_{Nx} = \bigwedge_M \left(r_{MN} \to \bigvee_{y \in x} f_{My} \right) \wedge \bigwedge_{y \in x} \bigvee_M (r_{MN} \wedge f_{My})$$

(here $\bigwedge, \bigvee, \wedge, \to$ are the operations of $\mathcal{C}'$). [Show this by proving that the left-hand side $\in H$ iff the right-hand side $\in H$, for any H.] It follows by $\in$-induction on x ($\in A$) that $\|\boldsymbol{F}(\check{N}) = \check{x}\| = f_{Nx} \in \mathcal{C}_1$ for all $N < \Omega$. Now consider any $p_2 \in P_2$ and put $N_0 = \mathrm{dom}(p_2)$:

$$\|\check{p}_2 \in \boldsymbol{G}_2\| = \|\check{p}_2 \subseteq \boldsymbol{F}\| = \bigwedge_{N < N_0} \|\check{p}_2(\check{N}) = \boldsymbol{F}(\check{N})\|$$

$$= \bigwedge_{N < N_0} f_{N, p_2(N)} .$$

Thus $\|\check{p}_2 \in \boldsymbol{G}_2\| \in \mathcal{C}_1$ for all $p_2 \in P_2$.

We shall now use $\boldsymbol{S}$ to get $\boldsymbol{G}_1$. Recall that $\bar{b} = \langle b_i \mid i \in A\rangle$, range$(\bar{b}) = \mathcal{B}_0$. Consider any $p_1 \in P_1$ and choose $i \in A$ such that $p_1 = b_i = \bar{b}(i)$. Since $\|\boldsymbol{F} : \check{\Omega} \xrightarrow{\text{onto}} \check{A}\| = 1$ we have:

$$\|\check{p}_1 \in \boldsymbol{G}_1\| = \|\check{\bar{b}}(\check{i}) \in \boldsymbol{G}_1\|$$

$$= \|(\exists N < \check{\Omega})\ [\check{i} = \boldsymbol{F}(N)\ \&\ \check{\bar{b}}(\boldsymbol{F}(N)) \in \boldsymbol{G}_1]\|$$

$$= \|(\exists N < \check{\Omega})\ [\check{i} = \boldsymbol{F}(N)\ \&\ N \in \boldsymbol{S}]\|$$

$$= \bigvee_{N < \Omega} (f_{Ni} \wedge s_N) \qquad (\bigvee = \bigvee^{\mathcal{C}'}, \wedge = \wedge^{\mathcal{C}'}).$$

Since $f_{Ni} \in \mathcal{C}_1 \subseteq \mathcal{C}_2$ and $s_N \in \mathcal{C}_2$ we see that $\|\check{p}_1 \in \boldsymbol{G}_1\| \in \mathcal{C}_2$ for all $p_1 \in P_1$, and by the above $\|\check{p}_2 \in \boldsymbol{G}_2\| \in \mathcal{C}_1 \subseteq \mathcal{C}_2$ for all $p_2 \in P_2$. Thus

$$\|(p_1, p_2) \in \boldsymbol{G}\| = \|\check{p}_1 \in \boldsymbol{G}_1\ \&\ \check{p}_2 \in \boldsymbol{G}_2\| \in \mathcal{C}_2$$

for all $(p_1, p_2) \in P_1 \times P_2$. But $\|(p_1, p_2) \in G\| = \|(p_1, p_2) \in H\| = (p_1, p_2)$ and thus $P_1 \times P_2 \subseteq \mathcal{C}_2$. Since $P_1 \times P_2$ is dense in $\mathcal{C}'$, $\mathcal{C}_2 = \mathcal{C}'$ and thus

$$\mathcal{C}' = [\{r_{MN} \mid M,N < \Omega\} \cup \{s_N \mid N < \Omega\}]^{<\infty}$$

(this holds both in the real universe and in $\mathfrak{M}$), completing the proof of (II), and of the whole theorem. □

26.9. Remark. (1) The segment of the proof of (I) in which it was shown that $\|\check{p}_2 \in G_2\| \in \mathcal{C}_1$ for all $p_2 \in P_2$ is actually (with trivial modifications) the proof that the B.a. RO(P_2) in which P_2 is dense is $\leqslant\Omega$-generated. Similarly, the proof of (I) shows that RO(P_2) (which is isomorphic to RO(W) discussed near the end of §25) is $(<\Omega, <\infty)$-distributive (in §25 we have quoted this from [19]).

(2) Our proof of Kripke's theorem (the case $\Omega = \aleph_0$ of 26.1) has the following in common with Kripke's proof ([8, §19]; the exposition in [12] eliminates the use of forcing in the proof, but does not change the countably-generated B.a. itself): To prove that the theorem holds in $\mathfrak{M}$ one finds a Cohen extension $\mathfrak{N}$ of $\mathfrak{M}$, obtained by adjoining to a subset of, or relation on ω (in our proof $R \subseteq \omega \times \omega$, $S \subseteq \omega$), such that in $\mathfrak{N}$ there is an $\mathfrak{M}$-generic u.f. on the B.a. $\mathcal{B}_0 \in \mathfrak{M}$ which one wants to embed. In Kripke's proof one makes the set of subsets of $\mathcal{B}_0$ which have a join in $\mathcal{B}_0$ and belong to $\mathfrak{M}$ countable in $\mathfrak{N}$, and then uses the Rasiowa–Sikorski Lemma in $\mathfrak{N}$. Thus if $|\mathcal{B}_0| = \kappa$ in $\mathfrak{M}$ one needs the collapsing $(\aleph_0, 2^\kappa)$ B.a. In our proof we collapse only κ itself to $\aleph_0$, but at the same time add an $\mathfrak{M}$-generic u.f. on $\mathcal{B}_0$ directly. Thus our procedure is more "economic", in a sense which can be made precise by cardinality estimates.

There is another Kripke-type theorem in the literature, namely the theorem of [15, 2.3], whose proof seems to be motivated by a similar argument, except that instead of our simultaneous adjunction to $\mathfrak{M}$ of $\mathfrak{M}$-generic filters G_1 on P_1 and G_2 on P_2, one needs there an iterated Cohen extension (as studied in [24]), so that the resulting B.a. is somewhat more complex.

We conclude this section by relating the B.a. $\mathcal{C}'$ described in 26.4 (which is well-defined up to isomorphism given $\Omega, A, \mathcal{B}_0, \langle b_i \mid i \in A\rangle$) with the B.a. $\mathcal{C}$ of cylindric functions studied in §21. Recall that, putting $W = \{x \mid x : \Omega \to A\}$, $\mathcal{C}$ is the subalgebra of $\mathcal{B}_0^W$ consisting of the functions $\xi : W \to \mathcal{B}_0$ which depend on $<\Omega$-coordinates. We will now prove that $\mathcal{C}'$ is isomorphic to the normal completion of $\mathcal{C}$ (assuming Ω regular, A transitive, $\mathcal{B}_0 = \{b_i \mid i \in A\}$ non-degenerate).

Since $\mathcal{C}'$ is complete it suffices to show that $\mathcal{C} \sim \{0\}$ has a dense sub-poset isomorphic to $P_1 \times P_2$ (where $P_1 = \mathcal{B}_0 \sim \{0\}$, say, and P_2 is defined in 26 4).

Recall from §20 the functions $\tilde{\phi}_{p,i} : W \to \mathcal{B}_0$ ($p \in P_2$, $i \in A$) given by

$$\tilde{\phi}_{p,i}(x) = \begin{cases} b_i & p \subseteq x, \\ 0 & \text{otherwise}, \end{cases} \quad (x \in W) .$$

As shown there, $\{\tilde{\phi}_{p,i} \mid p \in P_2, i \in A\}$ is a dense subset of $\mathcal{C}$. Therefore it suffices to show that $\{\tilde{\phi}_{p,i} \mid p \in P_2, i \in A, \tilde{\phi}_{p,i} \neq 0\}$, partially ordered by $\leqslant^{\mathcal{C}}$, is isomorphic to $P_1 \times P_2$. However, $\tilde{\phi}_{p,i} \neq 0$ iff $b_i \neq 0$, and assuming $b_i \neq 0$ $\tilde{\phi}_{p,i} \leqslant^{\mathcal{C}} \tilde{\phi}_{p',i'}$ if $(\forall x \in W)$ $[p \subseteq x \Rightarrow p' \subseteq x \;\&\; b_i \leqslant b_{i'}]$ iff $p \supseteq p'$ and $b_i \leqslant b_{i'}$. It follows immediately that the mapping $\phi_{p,i} \mapsto (b_i, p)$ ($p \in P_2$, $i \in A$, $b_i \neq 0$) is well-defined and is an isomorphism between a dense subset of $\sim \{0\}$ and $P_1 \times P_2$ ($P_1 = \mathcal{C}_0 \sim \{0\}$), completing the proof.

A simple computation which we omit shows that not only is $\mathcal{C}'$ isomorphic to the normal completion of $\mathcal{C}$, but in the natural isomorphism the $\leqslant\Omega$ generators r_{MN}, s_N of $\mathcal{C}'$ become the natural generators of (i.e., the cylindric functions corresponding to atomic formulas).

Since a regular subalgebra of a $(<\Omega, <\infty)$-distributive B.a. is $(<\Omega, <\infty)$-distributive itself, we see that the B.a. $\mathcal{C}$ of §21 and its dense subalgebras needed for Theorem 22.1 (in the case $\aleph_\alpha = \Omega$ at least) are $(<\Omega, <\infty)$-distributive if the initial B.a. $\mathcal{B}_0$ (called there Q) is. This gives us the embedding theorem for $(<\Omega, <\infty)$-distributive B.a.'s with cardinality estimates, but we have no space for a detailed discussion of it.

§27. Concluding remarks and an application

The reader will notice that there are many elaborations and details which we have left out mainly because they are rather similar to the case of B.a.'s discussed in detail in §1–18. He may therefore wish to know which are the truly open questions or unexplored directions to which the present work leads.

One such direction is mentioned at the end of §24, clarifying what happens to the relation ≡ between lattice terms when one considers a class Λ of p.o.s's which does not satisfy assumption (I) of §24 and what can be said from the aspects we have been studying about free $<\kappa$-complete p.o.s's etc. w.r.t. the class Λ (is there only one natural definition of $\mathcal{F}^{<\kappa}(\nu)$ in such cases?) It is also not clear whether some natural classes satisfy (I) or not.

Another direction is that of finding which distributive laws are preserved in the transition from Q to $\mathcal{C}$ (§§20–21). This is not yet completely clear even for B.a's.

One can also investigate which other properties are preserved in the transition from Q to $\mathcal{C}$, and also ask what happens if one want embeddings in complete lattices (or p.o.s's in general), when the class under consideration is not closed under normal completions.

This is only a small sample of the variety of problems one can work on, and I cannot say at present which ones will prove the most fruitful. There are also other, less typically algebraic, aspects of the work which can be developed, like the provability of embedding theorems in weak set theories, the realization of existence assertions by "computable" set functions and perhaps functorial aspects of the constructions.

Here is an example of the application of our work to models of weak set theories. We shall assume acquaintance with the primitive-recursive set-functions of [9], and call a set M *prim.-rec. closed* when it is closed under these functions. M is called *locally countable* (the term is due to M. Nadel) when each $x \in M$ has a $1-1$ mapping f into ω such that $f \in M$. By a *special* B.t. we mean a Boolean term supported by ω (i.e., on the variables p_n, $n < \omega$).

27.1. Theorem. *Let M be a prim.-rec. closed set satisfying the following condition: If ϕ is a special B.t., $\phi \in M$ and ϕ has a Boolean model in M* (i.e., *there is a valuation $(\mathfrak{B}, \Gamma) \in M$ in which ϕ is defined and $\|\phi\| > 0$*), *then ϕ has a two-valued model in M* (i.e., *there is a valuation as above with $\mathfrak{B} = \bar{2} =$ the two element B.a.* $\{0 \leqslant 1\}$). *Then M is locally countable.*

27.2. Remark. Conversely, if M is prim.-rec. closed and locally countable, then M (is transitive and) satisfies the above condition for all (not only special) B.t's, but the proof of this is easy by well-known methods and not related to this work. Both 27.1 and its converse are part of a general study of infinitary logic on prim.-rec. closed sets, which we pursue elsewhere.

Proof of Theorem 27.1 (outlined). If M consists only of finite sets it is clearly locally countable. Therefore assume $\omega \in M$. Let a be any transitive non-empty set, $a \in M$. Consider the predicate sentence

$$\phi_a = \bigwedge_{x \in a} (\exists u)(\pi_x(u))$$

(π_x being the locating formulas of 2.1). Denote $Z = \{v_n \mid n < \omega\}$. By the method of 5.3 extend $Y_0 = \{\phi_a\}$ to a set T of formulas so that 5.1(α), (β) hold for Z and T. Define the relation $\overset{*}{\to}$ on T by: $\phi \overset{*}{\to} \psi$ iff the formula $\phi \to \psi$ is satisfied in the model $\langle a, \in\rangle$ for all assignment (from its finitely many free variables into a). Form $\mathcal{B} = T/\overset{*}{\to}$ as in 5.1 and note that since $\langle a, \in\rangle \models \phi_a$ (by (2.2)), the equivalence class $[\phi_a]$ is $1^{\mathcal{B}}$ and $1^{\mathcal{B}} \neq 0^{\mathcal{B}}$. Now consider the B.t. $\sigma_c = \mathrm{QE}(\phi_a, Z)$ where $Z = \{v_n \mid n < \omega\}$ and QE is defined in 12.1. In other words $\sigma_a = \bigwedge_{x \in a} \bigvee_{n<\omega} \pi_x^n$ where the B.t's π_x^n are defined in 12.3 putting $\Omega = \omega$, $q_{mn} = p_{v_m \in v_n}$ $(m, n < \omega)$. Define I as in 12.2 and then by 12.2 $\|\sigma_a\|_{\mathcal{B},I} = [\phi_a] = 1^{\mathcal{B}} \neq 0^{\mathcal{B}}$.

The next thing to notice is that all objects constructed thus far (ϕ_a, Z, T, $\overset{*}{\to}$, B, σ_a, I) are members of M because $a, \omega \in M$ and M is prim.-rec. closed. Also, σ_a is a B.t. on the variables q_{mn} $(m, n < \omega)$ and hence can be made special in M by renaming its variables. Therefore, since σ_a has a Boolean-valued model $(\mathcal{B}, I)$ in M, it has a two-valued one $(\bar{2}, I_0) \in M$. Define $f: a \to \omega$ by: $f(x) =$ smallest $n < \omega$ such that $I_0 \models \pi_x^n$ ($I_0 \models \psi$ is short for: $\|\psi\|_{\bar{2}, I_0} = 1$). f is well-defined because $I_0 \models \bigwedge_{x \in a} \bigvee_{n<\omega} \pi_x^n$, and $f \in M$ by the closure properties of M. Also f is $1-1$ because for $x \neq y$, $I_0 \not\models \pi_x^n \wedge \pi_y^n$ (see 13.1). Thus a is countable in M.

Since a is an arbitrary transitive non-empty set in M, it is easy to conclude that M is locally countable. □

Similarly we could show that in the theorems of Gaifman–Hales and Kripke as given in 5.5 and 5.8, the objects whose existence is asserted can be obtained primitive-recursively from the given objects. Note that in this context the syntactical approach of Ch. I, especially §5 and §12, is more valuable than the apparently more elegant approach via cylindric functions which depends on the space W of *all* assignments.

Added in proof (June, 1975)

(1) Since the completion of this paper Conjecture 5.7 has been proved. For the proof and related results see [26a & b].

(2) Following the proof of 9.5 it is stated that the statement "if $\mathfrak{B}$ is a complete $(\aleph_0, <\aleph_1)$-generated B.a., then $CC(\mathfrak{B}) \leqslant \aleph_1$" is independent of ZFC. Here is a sketch of the proof:

Assume Martin's axiom and $2^{\aleph_0} > \aleph_1$. Let $A \subseteq 2^\omega$, $|A| = \aleph_1$. By a lemma of Silver [15, § 2.5] every subset of A is G_δ relative to A, hence Borel relative to A. Thus the B.a. $\mathfrak{B}$ of all subsets of A is generated in the $<\aleph_1$-sense by $\{b_n | n < \omega\}$ where $b_n = \{x \in A | x(n) = 1\}$ for each n. $\mathfrak{B}$ is a complete $(\aleph_0 < \aleph_1)$-generated B.a. with $\aleph_1$ disjoint elements.

(3) The fact about weakly compact cardinals stated following Theorem 9.6 has also been observed (independently) by Jech, and a proof can be found in [8a, p. 5]. The question whether there exists a complete $\aleph_0$-generated B.a. of cardinality κ when κ is strongly Mahlo but not weakly compact is, to the author's knowledge, still open.

(4) In 14.2 it seems as if we are not proving the claim for all large enough κ but only for all large enough κ of cofinality $\geqslant \mu$. However, if we let $\kappa_0 = \min \{\kappa | \mu \leqslant \mathrm{cf}(\kappa), \nu \leqslant 2^{<\kappa}\}$ then a trivial modification of the proof shows that the conclusion holds for all $\kappa \geqslant \kappa_0$. The same remark applies to 16.6.

(5) For further developments and applications of the ideas of § 18 see [26]. All three works [25, 26, 26a] have in fact grown from the present paper.

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